HOMEWORK #4, MATH 223, SPRING 2019

JOEL FRIEDMAN

Part of the homework will use the *inclusion-exclusion principle*; a special case of this principle says that if A_1, A_2 are subsets of a finite set, then

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| \tag{1}$$

where use the notation $|B|$ to denote the cardinality (size) of a set B. Similarly, if A_1, A_2, A_3 are subsets of a finite set, then

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|;$$

the principle generalizes in many ways, such as to any number of subsets of a set whose elements are *weighted* with appropriate finiteness conditions on the weights.

The textbook proves that for finite dimensional subspaces V_1, V_2 of a vector space V, we have

$$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2), \tag{2}$$

where $V_1 + V_2$ is defined to be

$$V_1 + V_2 = \bigl\{ v_1 + v_2 \in V \;\bigm|\; v_1 \in V_1 \text{ and } v_2 \in V_2 \bigr\} \tag{3}$$

(one easily checks that this set is a subspace of V).

The following theorem from class and the textbook is used to prove everything in Sections 3.1–3.4, including (2).

Theorem 0.1 (Basis Extension Theorem)**.** *Let V be a finite dimensional (real) vector space, i.e., one spanned by a finite set of vectors. Let $v_1, \ldots, v_s \in V$ be linearly independent, and $u_1, \ldots, u_t \in V$ such that*

$$V = \mathrm{Span}(v_1, \ldots, v_s, u_1, \ldots, u_t). \tag{4}$$

Then V has a basis consisting of $v_1, \ldots, v_s$ and some elements of $\{u_1, \ldots, u_t\}$.

Research supported in part by an NSERC grant.

Homework Problems

(1) Let V_1, V_2 be subspaces of a finite dimension vector space V. The point of this exercise to prove (2) in a bit more detail than is done in the textbook.

(a) Explain in **20–50 words**, and **using the Basis Extension Theorem**, why any basis, $v_1, \ldots, v_r$, of $V_1 \cap V_2$ can be extended to a basis, $v_1, \ldots, v_r, u_1, \ldots, u_s$, of V_1.

(b) Explain in **6–15 words** why **given your solution to part (a)**, any basis, $v_1, \ldots, v_r$, $V_1 \cap V_2$ can be extended to a basis, $v_1, \ldots, v_r, w_1, \ldots, w_t$, of V_2.

(c) Explain **in 8-20 words** what is the dimension of $V_1 \cap V_2$ in terms of r, s, t, using facts shown above.

(d) Explain briefly what are the dimensions of V_1 and of V_2 in terms of r, s, t, using facts shown above.

(e) With notation in parts (a) and (b), let

$$B = \{v_1, \ldots, v_r, u_1, \ldots, u_s, w_1, \ldots, w_t\}.$$

(i) Explain in **6–15 words** why each u_i $(i = 1, \ldots, s)$ lies in $V_1 + V_2$ (i.e., can be written as some element of V_1 plus some element of V_2).

(ii) Explain in **6–15 words** why each w_i $(i = 1, \ldots, t)$ lies in $V_1 + V_2$ (i.e., can be written as some element of V_1 plus some element of V_2).

(iii) Explain in **9–25 words** why each v_i $(i = 1, \ldots, r)$ lies in $V_1 + V_2$ (i.e., can be written as some element of V_1 plus some element of V_2); then explain why $\mathrm{Span}(B)$ is contained in $V_1 + V_2$.

(iv) Prove that the elements of B are linearly independent by showing that if

$$\sum_{i=1}^{r} \alpha_i v_i + \sum_{j=1}^{s} \beta_j u_j + \sum_{k=1}^{t} \gamma_k w_t = 0$$

then all the $\alpha_i, \beta_j, \gamma_k$ must be 0. [Hint: one way of doing this is to write

$$\sum_{i=1}^{r} \alpha_i v_i + \sum_{j=1}^{s} \beta_j u_j = -\sum_{k=1}^{t} \gamma_k w_t$$

and to explain in 20–60 words why $u = 0$ and hence all the β_j must be 0. Then one can finish in another 10-25 words.]

(f) Briefly explain why B spans all of $V_1 + V_2$.

(g) Using the above, briefly explain why B is a basis for $V_1 + V_2$.

(h) Using the above, briefly explain what is the dimension of $V_1 + V_2$ in terms of r, s, t, using facts shown above.

(2) Let

$$V_1 = \{f \colon \mathbb{Z} \to \mathbb{R} \mid f(n+4) = f(n)\}, \quad V_2 = \{f \colon \mathbb{Z} \to \mathbb{R} \mid f(n+5) = f(n)\}.$$

(a) Briefly explain: what are the dimesions of V_1 and V_2?

(b) Briefly explain: what is a simple description of $V_1 \cap V_2$?

(c) Briefly explain: what is the dimension of $V_1 + V_2$?

(3) Same question as the previous exercise, except with

$$V_1 = \{f\colon \mathbb{Z} \to \mathbb{R} \mid f(n+8) = f(n)\}, \quad V_2 = \{f\colon \mathbb{Z} \to \mathbb{R} \mid f(n+10) = f(n)\}.$$

(4) Fix an $n \in \mathbb{Z}$, and let $e_1, \ldots, e_n$ denote the standard basis of $\mathbb{R}^n$. For $A \subset \{1, \ldots, n\}$, let E_A denote the span of all e_a with $a \in A$. Let A_1, A_2 be two subsets of $\{1, \ldots, n\}$.

(a) What is the dimension of E_A in terms of $|A|$? Briefly justify your answer.

(b) Describe $E_{A_1} \cap E_{A_2}$, and briefly justify your answer. What is its dimension?

(c) Describe $E_{A_1} + E_{A_2}$, and briefly justify your answer. What is its dimension?

(d) If $V_1 = E_{A_1}$ and $V_2 = E_{A_2}$, use your answers to give an expression for

$$\dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2),$$

an expression for

$$\dim(V_1 + V_2),$$

and explain why inclusion-exclusion implies that these are equal.

(e) Let A_1, A_2, A_3 be subsets of $\{1, \ldots, n\}$, and set $V_i = E_{A_i}$ for $i = 1, 2, 3$. Show that

$$\dim(V_1+V_2+V_3) = \dim(V_1)+\dim(V_2)+\dim(V_3)-\dim(V_1\cap V_2)-\dim(V_1\cap V_3)-\dim(V_2\cap V_3)+\dim(V_1\cap V_2\cap V_3).$$

(f) If V_1, V_2, V_3 are subspaces of $\mathbb{R}^n$, is it always true that

$$\dim(V_1+V_2+V_3) = \dim(V_1)+\dim(V_2)+\dim(V_3)-\dim(V_1\cap V_2)-\dim(V_1\cap V_3)-\dim(V_2\cap V_3)+\dim(V_1\cap V_2\cap V_3)\ ?$$

Department of Computer Science, University of British Columbia, Vancouver, BC V6T 1Z4, CANADA, and Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, CANADA.

E-mail address: jf@cs.ubc.ca or jf@math.ubc.ca

URL: http://www.math.ubc.ca/~jf