SOLUTIONS TO HOMEWORK #2, MATH 223, SPRING 2019

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Homework Problems

3.10 Verify that
$$\sum_{m=1}^{n} m^2 = \binom{n+1}{2} + 2\binom{n+1}{3}$$
gives our usual formula for $\sum_{m=1}^{n} m^2$.

Solution:
$$\binom{n+1}{2} + 2\binom{n+1}{3} = \frac{(n+1)n}{2} + 2\frac{(n+1)n(n-1)}{6}$$
$$= (n+1)n\frac{3+2(n-1)}{6} = (n+1)n(2n+1)/6,$$
which is our familiar expression for $\sum_{m=1}^{n} m^2$.

2.1 Let $\mathcal{L}$ be the map taking a differentiable function, f, to the function $\mathcal{L}f$ defined by
$$\mathcal{L}f = \frac{d}{dx}f - 3f.$$

(a) Show that for any $C \in \mathbb{R}$, $f(x) = Ce^{3x}$ lies in $\ker(\mathcal{L})$.
(b) Show that if $f \in \ker(\mathcal{L})$, then $g(x) \stackrel{\text{def}}{=} f(x)e^{-3x}$ satisfies $g'(x) = 0$ for all x.
(c) Show that if $f \in \ker(\mathcal{L})$, then $f(x)$ must be of the form Ce^{3x} for some $C \in \mathbb{R}$.
(d) Find a polynomial of degree one, $p(x) = a_0 + a_1 x$, such that $\mathcal{L}p = x$.
(e) Find all solutions to the equation $\mathcal{L}f = x$.

Solution:
(a) $f'(x) = C3e^{3x}$, so $f' - 3f = 0$, so $f \in \ker(\mathcal{L})$
(b) By the product rule, $g' = f'e^{-3x} + f(-3)e^{-3x} = (f' - 3f)e^{-3x} = 0$.

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(c) Since $g' = 0$ (for g above), $g(x) = C$ for a constant $C \in \mathbb{R}$, and hence $fe^{-3x} = g = C$ and so $f = Ce^{3x}$.

(d) We have

$$\mathcal{L}p = p' - 3p = a_1 - 3(a_0 + a_1 x) = (-3a_0 + a_1) - 3a_1 x.$$

Setting $\mathcal{L}p = x$ is equivalent to

$$(-3a_1)x + (a_1 - 3a_0) = x,$$

i.e.,

$$-3a_1 = 1, \quad a_1 - 3a_0 = 0;$$

the first displayed equation gives $a_1 = -1/3$; the second gives $a_0 = a_1/3 = -1/9$. In other words, $p(x) = -x/3 - 1/9$

(e) Since $\mathcal{L}f = x$ has a particular solution $p(x) = -x/3 - 1/9$, the general solution is $p(x)$ plus any element of $\ker(\mathcal{L})$, i.e.,

$$-x/3 - 1/9 + Ce^{3x}.$$

2.2 Let $\mathcal{L}$ be the map taking a function $f\colon \mathbb{Z} \to \mathbb{R}$ to the function $\mathcal{L}$ defined by

$$(\mathcal{L}f)(n) = f(n+1) - 2f(n).$$

Show that $f \in \ker(\mathcal{L})$ iff f is given as

$$f(n) = C2^n$$

for some $C \in \mathbb{R}$.

Solution: Assume that $f \in \ker(\mathcal{L})$, i.e., $f(n+1) = 2f(n)$ for all $n \in \mathbb{Z}$. Then

$$f(1) = 2f(0), \quad f(2) = 2f(1) = 2^2 f(0), \quad f(3) = \cdots = 2^3 f(0),$$

and similarly $f(n) = 2^n f(0)$ for all $n \in \mathbb{N}$. We similarly see that $f(0) = 2^n f(-n)$ for all $n \in \mathbb{N}$, and hence $f(n) = C2^n$ where $C = f(0)$.

Conversely, assume that $f(n)$ is given by $C2^n$ for a constant C. Then for all $n \in \mathbb{Z}$,

$$f(n+1) - 2f(n) = C2^{n+1} - C \cdot 2 \cdot 2^n = 0.$$

Hence $f \in \ker(\mathcal{L})$.

2.3 Let $\mathcal{L}_{\mathrm{Fib}}$ be the map taking a function $f\colon \mathbb{Z} \to \mathbb{R}$ to the function $\mathcal{L}_{\mathrm{Fib}} f$ defined by

$$(\mathcal{L}_{\mathrm{Fib}} f)(n) = f(n+2) - f(n+1) - f(n).$$

(a) Let $F\colon \mathbb{Z} \to \mathbb{R}$ be the *Fibonacci numbers*, given by
   (i) $F(1) = F(2) = 1$,
   (ii) $F(n) = F(n-1) + F(n-2)$ for $n \geq 3$,
   (iii) $F(n-2) = F(n) - F(n-1)$ for $n \leq 0$,

which yields the familiar sequence

$$\ldots 13, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

Show that $F \in \ker(\mathcal{L}_{\mathrm{Fib}})$.

(b) Show that for any $r \in \mathbb{R}$, the function $g\colon \mathbb{Z} \to \mathbb{R}$ given by $g(n) = r^n$ lies in $\ker(\mathcal{L}_{\mathrm{Fib}})$ iff r satifies
$$r^2 - r - 1 = 0.$$
(c) Let $\xi_+ = (1+\sqrt{5})/2$ and $\xi_- = (1-\sqrt{5})/2$. Show that for any $b_0, b_1 \in \mathbb{R}$ there are unique x, y with
$$\begin{aligned} x \;+\; y &= b_0 \\ \xi_+ x + \xi_- y &= b_1 \end{aligned}$$
(d) Explain why every element, f, of $\ker(\mathcal{L}_{\mathrm{Fib}})$ is uniquely determined by its values $f(0)$ and $f(1)$.
(e) Explain why every element, f, of $\ker(\mathcal{L}_{\mathrm{Fib}})$ is uniquely expressible as
$$f(n) = x\,\xi_+^n + y\,\xi_-^n$$
for some $x, y \in \mathbb{R}$ [Hint: show that there is a unique x, y satisfying this formula for $n = 0$ and $n = 1$.]
(f) Find a formula for the Fibonacci numbers, $F(n)$, with n above.

Solution:
(a) The condition $F(n) = F(n-1) + F(n-2)$ for all $n \in \mathbb{Z}$ is equivalent to $F(n+2) = F(n-1) + F(n)$ for all $n \in \mathbb{Z}$, which is equivalent to $F(n+2) - F(n-1) - F(n) = 0$ for all $n \in \mathbb{Z}$, which is equivalent to $F \in \ker(\mathcal{L}_{\mathrm{Fib}})$.
(b) We have $g \in \ker(\mathcal{L}_{\mathrm{Fib}})$ is equivalent to $r^{n+2} - r^{n+1} - r^n = 0$ for all $n \in \mathbb{Z}$; since $r = 0$ does not satisfy this equation for $n = 0$, we may add $r \neq 0$ to the latter of the equivalent statements. Hence both statements are equivalent to $r \neq 0$ and
$$r^n\big(r^2 - r - 1\big) = 0$$
for all $n \in \mathbb{Z}$; this is equivalent to $r \neq 0$ and
$$r^2 - r - 1 = 0,$$
which is equivalent to $r^2 - r - 1 = 0$ since $r = 0$ does not satisfy this equation.
(c) (There are many ways to do this; for example, we probably explained in class that you can "cross multiply" the coefficients and check that this is nonzero. Here is another way.) Multiplying the first equation by ξ_+ and subtracting the second equation from the first we get $(\xi_+ - \xi_-)y = \xi_+ b_0 - b_1$, and so the unique value for y is $y = (\xi_+ b_0 - b_1)/\sqrt{5}$. Next the top equation implies that $x = b_0 - y$. Since this is a 2×2 system, we see that for $b_0 = b_1 = 0$ we must have $y = x = 0$. Hence this 2×2 system has a unique solution for all x, y.
(d) Given $f(0), f(1)$, $f(2)$ is uniquely determined as $f(1) + f(0)$; $f(3)$ is uniquely determined as $f(2) + f(1)$; continuing in this fashion (or by induction), we see that for all $n \in \mathbb{N}$ we have that $f(n)$ is unique determined once $f(0), f(1)$ are fixed. Similarly $f(-1) = f(1) - f(0)$, $f(-2) = f(0) - f(-1)$, etc., and we see that for all $n \in \mathbb{Z}$, $f(-n)$ is uniquely determined once $f(0), f(1)$ are determined.

Conversely, given $f(2), f(3), \dots$ and $f(-1), f(-2), \dots$ determined in this way from $f(0)$ and $f(1)$, we have that

(1) $$f(2) - f(1) - f(0) = 0, \quad f(3) - f(2) - f(1) = 0,$$

(from the values given to $f(2), f(3), \dots$), and

(2) $$f(1) - f(0) - f(-1) = 0, \quad f(0) - f(-1) - f(-2) = 0,$$

(from the values given to $f(-1), f(-2), \dots$). Hence such a function f satisfies $f(n+2) - f(n+1) - f(n) = 0$ for $n \geq 2$ (by (1)) and for $n \leq 1$ (by (2)), and hence for all $n \in \mathbb{Z}$.

(e) Let $f \in \ker(\mathcal{L}_{\mathrm{Fib}})$. For fixed $x, y \in \mathbb{R}$,

(3) $$g(n) = g(x, y; n) = x\xi_+^n + y\xi_-^n$$

lies in $\ker(\mathcal{L}_{\mathrm{Fib}})$; by part (c), we can find x, y such that

$$\begin{array}{rcl} x + y &=& f(0) \\ \xi_+ x + \xi_- y &=& f(1) \end{array},$$

whereupon $g(0) = f(0)$ and $g(1) = f(1)$, and by part (d) $f(n) = g(n)$ for all $n \in \mathbb{Z}$; therefore for all $n \in \mathbb{Z}$,

$$f(n) = g(n) = x\xi_+^n + y\xi_-^n$$

(f) Setting $b_0 = F(0) = 0$ and $b_1 = F(1) = 1$ in part (c), the formulas in part (c) show that $y = (\xi_+ b_0 - b_1)/\sqrt{5} = -1/sqrt5$ and $x = b_0 - y = 1/\sqrt{5}$. Hence

$$F(n) = \xi_+^n/\sqrt{5} - \xi_-^n/\sqrt{5}.$$

2.4 Let $\mathcal{L}$ be the operator taking a function $f\colon \mathbb{Z} \to \mathbb{R}$ to the function $\mathcal{L}f$ defined by

$$(\mathcal{L}f)(n) = f(n+2) - f(n).$$

(a) Show that $f \in \ker(\mathcal{L})$ iff f is of the form

$$f(n) = \begin{cases} a, & \text{if } n \text{ is even, and} \\ b, & \text{otherwise.} \end{cases}$$

for some $a, b \in \mathbb{R}$.

(b) Show that for any $r \in \mathbb{R}$, the function $g\colon \mathbb{Z} \to \mathbb{R}$ given by $g(n) = r^n$ lies in $\ker(\mathcal{L})$ iff r satifies

$$r^2 - 1 = 0.$$

(c) Show that $f \in \ker(\mathcal{L})$ iff f is of the form

$$f(n) = x + (-1)^n y$$

for some $x, y \in \mathbb{R}$.

Solution: This is the same idea(s) as Problem 2.3 above, but easier as far as the calculations go.

(a) $f \in \ker(\mathcal{L})$ iff $f(n+2) = f(n)$ for all $n \in \mathbb{Z}$; this implies

$$\cdots = f(-4) = f(-2) = f(0) = f(2) = f(4) = \cdots$$

$$\cdots = f(-3) = f(-1) = f(1) = f(3) = \cdots,$$

and these two conditions clearly imply $f(n+2) = f(n)$ for all $n \in \mathbb{Z}$; the two conditions are equivalent to the desired conditions.

(b) Reasoning at in Problem 2.3, $g(n+2) - g(n)$ is equivalent to $r^n(r^2-1) = 0$ for all $n \in \mathbb{Z}$, which implies that $r \neq 0$ and is therefore equivalent to $r^2 - 1 = 0$ (whose solutions are $r = \pm 1$).

(c) Reasoning at in Problem 2.3, f is determined by its values $f(0), f(1)$, and any function of the form $g(n) = x1^n + y(-1)^n = x + y^(- 1)^n$ lies in $\ker(\mathcal{L})$. Since

$$x + y = f(0), \quad x - y = f(1)$$

has a unique solution (x, y), for any given $f(0), f(1)$, for this value of (x, y) we have $f(n) = g(n) = x + y^(- 1)^n$ for all $n \in \mathbb{Z}$.

2.5 Let $\mathcal{L}$ be the operator taking a function $f\colon \mathbb{Z} \to \mathbb{R}$ to the function $\mathcal{L}f$ defined by

$$(\mathcal{L}f)(n) = f(n+4) - f(n).$$

We similarly define $\mathcal{L}$ on functions $\mathbb{Z} \to \mathbb{C}$.

(a) Say that $f \in \ker(\mathcal{L})$ and $f(0) = 0$, $f(1) = 1$, $f(2) = 2$, and $f(3) = 3$. Describe $f(n)$ for all n.

(b) Show that for any $r \in \mathbb{C}$, the function $g\colon \mathbb{Z} \to \mathbb{C}$ given by $g(n) = r^n$ lies in $\ker(\mathcal{L})$ iff r satifies

$$r^4 - 1 = 0.$$

(c) Show that the solutions to $r^4 - 1 = 0$ are given by $r = 1, -1, i, -i$ where $i \in \mathbb{C}$ denotes $\sqrt{-1}$.

(d) Show that $f \in \ker(\mathcal{L})$ iff f is of the form

$$f(n) = \alpha + \beta i^n + \gamma(-1)^n + \delta(-i)^n$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{C}$.

Solution: The main point of this problem is to show why complex numbers can be useful; the techniques are mostly like those of the previous two problems.

Parts (a) and (b) are done by reasoning as in the previous two problems. Part (c) is done by verifying that each of $r = 1, -1, i, -i$ solves $r^4 - 1 = 0$, and then noting that a polynomial of degree 4 has at most 4 roots. Part (d) follows as soon as we can convince ourselves that for any given $f(0), \ldots, f(3) \in \mathbb{C}$, the 4×4 system

$$\begin{array}{ccccccccc}
\alpha & + & \beta & + & \gamma & + & \delta & = & f(0) \\
\alpha & + & i\beta & - & \gamma & - & i\delta & = & f(1) \\
\alpha & - & \beta & + & \gamma & - & \delta & = & f(2) \\
\alpha & - & i\beta & - & \gamma & + & i\delta & = & f(3)
\end{array}$$

has a unique solution $(\alpha, \beta, \gamma, \delta)$. Adding all four equations together we get

$$4\alpha + 0\beta + 0\gamma + 0\delta = f(0) + f(1) + f(2) + f(3),$$

which uniquely determines α. If we multiply the first equation by 1, the second by i, the third by -1, and the fourth by $-i$, we get

$$\begin{array}{rcccccccc} \alpha & + & \beta & + & \gamma & + & \delta & = & f(0) \\ i\alpha & - & \beta & + & i\gamma & + & \delta & = & if(1) \\ -\alpha & + & \beta & - & \gamma & + & \delta & = & -f(2) \\ -i\alpha & - & \beta & - & i\gamma & + & \delta & = & -if(3) \end{array}$$

and adding these together similarly gives

$$4\delta = f(0) + if(1) - f(2) - if(3)$$

Again multiplying the first equation by 1, the second by i, the third by -1, and the fourth by $-i$ gives

$$4\gamma = f(0) - f(1) + f(2) - f(3).$$

Similarly we get

$$4\beta = f(0) - if(1) - f(2) + if(3).$$

Hence there is a unique solution in $(\alpha, \beta, \gamma, \delta)$. (There are other ways of solving this system.) **Solving this system illustrates what is called the *DFT (discrete Fourier transform)*, and is algorithmically solved by what is often called the *FFT (fast Fourier transform)*; this idea has numerous applications, such as multiplying two degree n polynomials quickly (for n large enough that the naïve multiplication algorithm is too slow).**

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