SOLUTIONS TO HOMEWORK #1, MATH 223, SPRING 2019

JOEL FRIEDMAN

Homework Problems

3.2 (a) We say that a polynomial $p(x) = a_0 + a_1x + a_3x^2 + a_3x^3$ is *odd* if $p(-x) = -p(x)$. For which a_0, a_1, a_2, a_3 is p odd?

> **Solution:** If $p(-x) = -p(x)$ then
>
> $a_0 + a_1(-x) + a_3(-x)^2 + a_3(-x)^3 = -(a_0 + a_1x + a_3x^2 + a_3x^3)$;
>
> expanding we see that the above equality is equivalent to
>
> $$2a_0 + 2a_2x^2 = 0$$
>
> (as polynomials). Since the LHS (left-hand-side) equals the zero polynomial, we have that $a_0 = 0$ and $a_2 = 0$. Hence p is odd for arbitrary $a_0, \ldots, a_3$ for which $a_0 = 0$ and $a_2 = 0$, i.e., $p(x) = a_1x + a_3x^3$.

(b) Show that if $p(x) = a_0 + a_1x + a_3x^2 + a_3x^3$ is odd, then $p(0) = 0$.

> **Solution:** By the last part $p(x) = a_1x + a_3x^3$, so $p(0) = a_1 \cdot 0 + a_3 \cdot 0 = 0$.

(c) If $p(x) = a_0 + a_1x + a_3x^2 + a_3x^3$, and $q(x) = p(x - 1/2)$ is odd, what can you say about [OLD: $q(-1/2)$?] the value of $q(0) = p(-1/2)$? How does this relate to the discussion in this subsection?

> **Solution:** [The problem as written had a error; the correction is given in red.] Since q is odd, $q(0) = 0$, and hence $p(-1/2) = q(0) = 0$. This relates to the above, since $p_2(n)$ is a polynomial of degree 3 and has $p_2(-1-n) = -p(n)$ for infinitely many n; it follows that $p(x)$ defined as $p_2(x - 1/2)$ has $p(-x) = -p(x)$ for infinitely value of x, and hence $p(-x) = -p(x)$ as polynomials. [This last point was explained in class: the point is that $p(-x) + p(x)$ is a polynomial, and since it has infinitely many roots it must be the zero polynomial; hence $p(-x) = -p(x)$ as polynomials.]

Research supported in part by an NSERC grant.

(d) We say that a polynomial $p(x) = a_0 + a_1x + a_3x^2 + a_3x^3$ is *even* if $p(-x) = p(x)$. For which a_0, a_1, a_2, a_3 is p even?

> **Solution:** Similarly to the above, p is even iff $a_1 = a_3 = 0$, i.e., iff $p(x) = a_0 + a_2x^2$.

(e) Show that if $p(x) = a_0 + a_1x + a_3x^2 + a_3x^3$ is even, then $p'(0) = 0$ where p' shorthand for the derivative dp/dx.

> **Solution:** From the previous part we have $p'(x) = 2a_2x$; hence $p'(0) = 2a_2 \cdot 0 = 0$.

3.3 If $f\colon \mathbb{Z} \to \mathbb{R}$ or $f\colon \mathbb{R} \to \mathbb{R}$, we say that

(a) f is *odd* if $f(-x) = -f(x)$ for all x (in the domain of f).

(b) f is *even* if $f(-x) = f(x)$ for all x (in the domain of f).

(a) Show that if $f\colon \mathbb{Z} \to \mathbb{R}$ or $f\colon \mathbb{R} \to \mathbb{R}$, then

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$

expresses f as the sum of an even plus an odd function; in other words, show that the first expression on the RHS (right-hand-side) is an even function, and second expression on the RHS is an odd function, and that the above equation is correct.

> **Solution:** Setting
> $$g(x) = \frac{f(x) + f(-x)}{2}$$
> then
> $$g(-x) = \frac{f(-x) + f(-(-x))}{2} = \frac{f(-x) + f(x)}{2} = g(x).$$
> Similarly setting $g(x) = \bigl(f(x) - f(-x)\bigr)/2$ we see that $g(-x) = -g(x)$. We easily verify the last part.

(b) If $f\colon \mathbb{Z} \to \mathbb{Z}$, is

$$\frac{f(x) + f(-x)}{2}$$

always a function $\mathbb{Z} \to \mathbb{Z}$? Either (1) show that it is, or (2) give a counterexample or show that it isn't always.

(c) Show that if f is odd, then $f(0) = 0$.

> **Solution:** The equation $f(-x) = -f(x)$ with $x = 0$ yields $f(0) = -f(0)$ and hence $2f(0) = 0$.

(d) Show that if $f\colon \mathbb{R} \to \mathbb{R}$ is odd and differentiable, then $f' = df/dx$ is even; show the same with "odd" and "even" exchanged.

> **Solution:** The chain rule shows that if $g(x) = f(-x)$ then $g'(x) = -f'(-x)$. Hence if f is odd, i.e., $f(-x) = -f(x)$,

then applying d/dx to both sides yields $-f'(-x) = -f'(x)$, i.e., $f'(-x) = f'(x)$, i.e., f' is even. Similarly f is even implies that f' is odd.

(e) Show that if f is odd and infinitely differentiable (i.e., has derivatives to all orders), then $f(0), f''(0), f''''(0), \ldots$ are zero. Similarly show that if $f : \mathbb{R} \to \mathbb{R}$ is even and infinitely differentiable, then $f'(0), f'''(0), \ldots$ are zero.

Solution: If f is infintely differentiable, then so are $f', f'', f''', \ldots$. The above shows that if f is odd, then f' is even, f'' is odd, etc. Hence $f, f'', f'''', \ldots$ are odd, and by the above their values at $x = 0$ are all 0. Similarly if f is even, then $f', f''', f^{(\mathrm{iv})}, \cdots$ are odd, and so their values at $x = 0$ are all 0.

(f) Show any function $\mathbb{Z} \to \mathbb{R}$ or $\mathbb{R} \to \mathbb{R}$ can be expressed *uniquely* as a sum of an even plus an odd function.

Solution: If f can be expressed as $g_1 + h_1$ and as $g_2 + h_2$ where g_1, g_2 are even and h_1, h_2 are even, then $g = g_1 - g_2$ is even and $h = h_2 - h_1$ is odd and $g = h$. Since g is even, we have $g(x) = g(-x)$, and since $g = h$ we have $h(x) = h(-x)$; but since h is odd we have $h(x) = -h(-x)$. It follows $h(-x) = h(x) = -h(-x)$ so $h = -h$ so $2h = 0$ (i.e. the zero function) so $h = 0$ (the zero function). Since $0 = h = h_2 - h_1$, we have $h_1 = h_2$. Since $g = h$ we have $g = h = 0$ and hence $g_1 = g_2$.

3.5 The binomial theorem (**??**) for $n = 4$ says that

$$(x + y)^4 = (x + y)(x + y)(x + y)(x + y) = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

Notice that there are four strings with three x's and one y:

$$xxxy,\ xxyx,\ xyxx,\ yxxx$$

and six strings with two x's and two y's:

$$xxyy,\ xyxy,\ xyyx,\ yxxy,\ yxyx,\ yyxx.$$

Notice that in both cases above we have listed the strings in *lexicographical order*, meaning the order they would appear in a dictionary (if they were words).

(a) List all strings of one x and three y's in lexicographical order.

Solution:

$$xyyy,\ yxyy,\ yyxy,\ yyyx$$

(b) List all strings of one x and four y's in lexicographical order.

Solution:

$$xyyyy,\ yxyyy,\ yyxyy,\ yyyxy,\ yyyyx$$

(c) List all strings of two x's and three y's in lexicographical order.

Solution:

$xxyyy$, $xyxyy$, $xyyxy$, $xyyyx$, $yxxyy$, $yxyxy$, $yxyyx$, $yyxxy$, $yyxyx$, $yyyxx$

(d) Using your answer to the last part, **describe—IN 15 WORDS OR FEWER—an algorithm** to list all strings of three x's and two y's in lexicographical order; i.e., **do not produce this list**, but instead describe **how** you would take the list you wrote in the last part **as input** and then **output** a list of all strings of three x's and two y's.

Solution: List the strings in reverse order and exchange the x's and y's.

(e) Explain how the number of elements in some of your lists above relate to the binomial theorem

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5y^4 + y^5.$$

Solution: When you write $(x+y)^5$ as

$$(x+y)\,(x+y)\,(x+y)\,(x+y)\,(x+y)$$

and expand naïvely (i.e., without collecting terms), you get 32 strings of x's and y's; when you collect terms you see the binomial coefficients.

3.7 Prove (**??**) (i.e., $\binom{1}{k} + \ldots + \binom{n}{k} = \binom{n+1}{k+1}$) directly, by noting that its right-hand-size represents the number of strings of $n-k$ x's and $k+1$ y's, and using the fact that each such string begins with some number of x's before it encounters its first y.

Solution: Each string, s, with $n-k$ x's and $k+1$ y's must have at least one y (we assume $k \geq 0$ so $k+1 \geq 1$); the string s must begin with some number of x's, say m of them ($m = 0, 1, \ldots$), followed by a y, then there are $n+1-(m+1) = n-m$ letters left, k of which must be y's. Hence

$$\binom{n+1}{k+1} = \sum_{m=0,1,\ldots} \binom{n-m}{k} = \binom{n}{k} + \binom{n-1}{k} + \cdots + \binom{k}{k};$$

we can add $\binom{k-1}{k}, \binom{k-2}{k}, \ldots, \binom{1}{k}$ to the RHS (right-hand-side) since they are zero.

3.9 Compute the function $(\mathcal{D}f)(n)$ for all $n \in \mathbb{N}$:

(a) $f(n) = (n-1)^2$;

Solution:

$$f(n+1) - f(n) = n^2 - (n-1)^2 = 2n - 1.$$

(b) $f(n) = (n-1)n(2n-1)/6$;

Solution:

$$f(n+1)-f(n)=\frac{n(n+1)(2n+1)}{6}-\frac{(n-1)n(2n-1)}{6}$$
$$=n\frac{(n+1)(2n+1)-(n-1)(2n-1)}{6}=n\frac{6n}{6}=n^2$$

(c) $f(n)=\binom{n}{4}\stackrel{\text{def}}{=}n(n-1)(n-2)(n-3)/24$;

Solution:

$$f(n+1)-f(n)=\frac{(n+1)n(n-1)(n-2)}{24}-\frac{n(n-1)(n-2)(n-3)}{24}$$
$$=n(n-1)(n-2)\frac{(n+1)-(n-3)}{24}=n(n-1)(n-2)\frac{4}{24}=\binom{n}{3}$$

(d) $f(n)=-(1/3)^{n-1}/2$ and simplify your answer.

Solution:

$$f(n+1)-f(n)=\frac{-(1/3)^n-(1/3)^{n-1}}{2}=-(1/3)^n\frac{1-3}{2}=(1/3)^n.$$

(e) Show how (**??**) (i.e., $(\mathcal{S}\mathcal{D}f)(n)=f(n+1)-f(1)$) and the above computations yield the following formulas:

$$1+3+5+\cdots+(2n-1)=n^2,$$
$$1+2^2+3^2+\cdots+n^2=n(n+1)(2n+1)/6,$$
$$\binom{1}{3}+\binom{2}{3}+\cdots+\binom{n}{3}=\binom{n+1}{4}$$
$$(1/3)^1+(1/3)^2+\cdots+(1/3)^n=\frac{1-(1/3)^n}{2},$$

Solution: They all arise from the formula

$$(\mathcal{S}\mathcal{D}f)(n)=f(n+1)-f(1);$$

for example, in the first case $f(n)=(n-1)^2$ we have

$$f(n+1)-f(1)=n^2$$

while

$$(\mathcal{S}\mathcal{D}f)(n)=\sum_{m=1}^{n}(\mathcal{D}f)(n)=\sum_{m=1}^{n}(2m-1)=1+3+5+\cdots+(2n-1),$$

and similarly for the other formulas.

Department of Computer Science, University of British Columbia, Vancouver, BC V6T 1Z4, CANADA, and Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, CANADA.

E-mail address: jf@cs.ubc.ca or jf@math.ubc.ca

URL: http://www.math.ubc.ca/~jf