Marks

[3] **1.** In which direction does the function $x^2 + 4y + xy$ have the maximum rate of change at the point $(-1, 6)$? What is this rate of change?

Answer: The gradient $f(x, y) = x^2+4y+xy$ **is** $\nabla f = \langle f_x, f_y\rangle = \langle 2x+y, 4+x\rangle$**, which at** $(-1, 6)$ **is** $\nabla f = \langle 2(-1)+6, 4+(-1)\rangle = \langle 4, 3\rangle$**. So the maximum rate of change is in the direction** $\nabla f/|\nabla f| = \langle 4/5, 3/5\rangle$**, and this rate of change is** $|\nabla f| = 5$**.**

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[6] **2.** Find and classify all the critical points of $x^3 - 3x + y^2$.

Answer: Setting $f = x^3 - 3x + y^2$ **we have** $\nabla f = \langle 3x^2 - 3, 2y \rangle$**. Setting this vector to zero gives** $3x^2 - 3 = 0 = 2y$ **so** $x = \pm 1$ **and** $y = 0$**. We have** $f_{xx} = 6x$**,** $f_{xy} = f_{yx} = 0$**,** $f_{yy} = 2$**, and hence**

$$D = f_{xx} f_{yy} - (f_{xy})^2 = 12x.$$

So at $(1, 0)$ **we have** $f_{xx} = 6$ **and** $D = 12$**, so** $(1, 0)$ **is a local minimum; at** $(-1, 0)$ **we have** $f_{xx} = -6$ **and** $D = -12$**, so** $(-1, 0)$ **is a saddle.**

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[6] **3.** Find the minimum and maximum values of $3x^2 - 4y^2 + 5z^2$ subject to $x^2 + y^2 + z^2 = 1$ using Lagrange multipliers.

Answer: If $f = 3x^2 - 4y^2 + 5z^2$ and $g = x^2 + y^2 + z^2 = 1$, the equation $\nabla f = \lambda \nabla g$ reads

$$\langle 6x, -8y, 10z \rangle = \lambda \langle 2x, 2y, 2z \rangle,$$

so

$$6x = 2\lambda x, \quad -8y = 2\lambda y, \quad 10z = 2\lambda z.$$

Since we have $x^2 + y^2 + z^2 = 1$, at least one of x, y, z must be nonzero. So if $x \neq 0$ we have $\lambda = 3$, and then $-8y = 6y$ so $y = 0$ and similarly $z = 0$. Since $x^2 + y^2 + z^2 = 1$ we have $x = \pm 1$, and $f(\pm 1, 0, 0) = 3$.

Similarly, if $y \neq 0$ then we have $\lambda = -4$ and $x = 0$, $z = 0$, and $y = \pm 1$; we have $f(0, \pm 1, 0) = -4$.

Similarly, if $z \neq 0$ then $\lambda = 5$, $x = y = 0$ and $z = \pm 1$, and $f(0, 0, \pm 1) = 5$.

It follows that the minimum value of f subject to $g = 1$ is -4, and its maximum value is 5.

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[6] **4.** Assume the twice differentiable function $F(x,y,z)$ satisfies the equation $F_z = F_{xx} + F_{yy}$. Let A be some constant and let $G(\gamma,s,t) = F(\gamma+s,\gamma-s,At)$ [the exam paper had t instead of At; this was corrected in both classrooms]. Find the value of A such that $G_t = G_{\gamma\gamma} + G_{ss}$ (for any F).

Answer: [See Homework 6 solutions, Problem 3 from Final 2012WT1.] By the chain rule we have

$$G_t = \frac{\partial G}{\partial t} = \frac{\partial}{\partial t} F(\gamma+s,\gamma-s,At)$$

$$= F_x(\gamma+s)_t + F_y(\gamma-s)_t + F_z(At)_t = AF_z.$$

Similarly

$$G_\gamma = F_x(\gamma+s)_\gamma + F_y(\gamma-s)_\gamma + F_z(At)_\gamma = F_x + F_y,$$

and

$$G_{\gamma\gamma} = (G_\gamma)_\gamma = (F_x)_\gamma + (F_y)_\gamma = (F_{xx}+F_{xy}) + (F_{yx}+F_{yy}).$$

Similarly $G_s = F_x - F_y$**, and**

$$G_{ss} = (F_{xx}-F_{xy}) - (F_{yx}-F_{yy}).$$

So

$$G_{\gamma\gamma} + G_{ss} = 2F_{xx} + 2F_{yy} = 2F_z,$$

by the equation for F**, so if** $G_{\gamma\gamma}+G_{ss} = 2F_z$ **must equal** $G_t = AF_z$**, then we must have** $A = 2$ **(since this has to hold for any** F**).**

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The End

Be sure that this examination has 6 pages including this cover

The University of British Columbia

Midterm Examinations - November 2015

Mathematics 200-102

Closed book examination

Time: 45 minutes

Name ______________________ **Signature** ______________________

Student Number ______________ **Instructor's Name** ______________

Section Number ______________

Special Instructions:

Calculators, notes, or other aids may not be used. Answer questions on the exam. This exam is two-sided!

Rules governing examinations

1. Each candidate should be prepared to produce his library/AMS card upon request.

2. Read and observe the following rules:

No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.

Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

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(a) Making use of any books, papers or memoranda, other than those authorized by the examiners.

(b) Speaking or communicating with other candidates.

(c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

3. Smoking is not permitted during examinations.

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