Marks

Exam 1: 1,5,9,13,17.
Exam 2: 2,6,10,14,18.
Exam 3: 3,7,11,15,19.
Exam 4: 4,8,12,16,20.

[1] **1.** The curve the Earth traces out relative to the sun most closely resembles: (A) a parabola, (B) an ellipse, (C) an elliptic paraboloid, (D) a circular paraboloid. Answer: **B**

[1] **2.** The surface of most common satellite dishes most closely resembles: (A) a parabola, (B) an ellipse, (C) an elliptic paraboloid, (D) a circular paraboloid. Answer: **D**

[1] **3.** The surface of the Earth most closely resembles: (A) a sphere, (B) an ellipsoid, (C) an elliptic paraboloid, (D) a circular paraboloid. Answer: **B**

[1] **4.** When the curve $z = x^2$ is rotated about the z-axis, what surface does it trace out? (A) a sphere, (B) an ellipsoid, (C) an elliptic paraboloid, (D) a circular paraboloid. Answer: **D**

[2] **5.** Find the centre of the sphere described by $x^2 - 4x + y^2 + 10y + z^2 - 12z = 200$.

Answer: We "complete the squares" to obtain the equivalent formula

$$(x-2)^2 + (y+5)^2 + (z-6)^2 = 200 + 2^2 + 5^2 + 6^2$$

which implies that the centre is $(2, -5, 6)$**.**

[2] **6.** Find the parametric form of the line $2x + 6 = 4y = z - 6$.

Answer: Setting $2x + 6 = 4y = z - 6 = t$ **and solving for** (x, y, z) **we have**

$$\langle x, y, z \rangle = \langle -3 + (1/2)t, (1/4)t, 6 + t \rangle \quad \textbf{or} \quad \langle -3, 0, 6 \rangle$$

Alternative method: we may rewrite these equations as

$$\frac{x+3}{1/2} = \frac{y}{1/4} = \frac{z-6}{1}$$

which (according to a formula on the formula sheet) gives the parametric from as $\langle -3, 0, 6 \rangle + t\langle 1/2, 1/4, 1 \rangle$**.**

[2] **7.** Find the area of the parallelogram whose sides point in the direction $\langle 1, 2, 1 \rangle$ and $\langle 1, 3, 1 \rangle$.

Answer: The area is the magnitude of $\langle 1, 2, 1 \rangle \times \langle 1, 3, 1 \rangle = \langle -1, 0, 1 \rangle$**, so equals** $|\langle -1, 0, 1 \rangle| = \sqrt{2}$**.**

Continued on page 3

[2] **8.** Find the volume of the parallelopided whose sides point in the direction $\langle 1,2,1\rangle$, $\langle 0,0,1\rangle$, and $\langle 0,4,3\rangle$.

Answer: The triple product of the vectors or—what is the same—the determinant whose rows are these vectors, is -4. Since the volume is the absolute value of this triple product or determinant, it equals 4.

[3] **9.** Today you are given a 3-dimensional vector, $\mathbf{a}$. Tomorrow you will be given 1,000 vectors, and you will want to compute the projection of these vectors onto $\mathbf{a}$. If tomorrow you want to use the fewest number of operations (additions, multiplications, etc.), what computation(s) should you do today and why? How many operations will you save? [Justify your answer with a formula on the formula sheet.]

Answer: Today you can normalize a via $\mathbf{u} = \mathbf{a}/|\mathbf{a}|$, which costs 3 divisions (either way you have to compute $|\mathbf{a}|$). Then the formula

$$\mathrm{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{b}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}$$

is replaced with

$$\mathrm{proj}_{\mathbf{a}}\mathbf{b} = \mathrm{proj}_{\mathbf{u}}\mathbf{b} = (\mathbf{b}\cdot\mathbf{u})\mathbf{u}$$

which tomorrow saves you one division (since $|\mathbf{u}| = 1$) for each of $1,000$ projection computations. Hence the total savings is $1,000$ minus the three divisions to find u, for a total savings of 997. Grading remark: one point for stating that you normalize a today to simplify the computation tomorrow, two points for the correct operation count.

[3] **10.** Today you are given a 3-dimensional vector, $\mathbf{a}$. Tomorrow you will be given 1,000 vectors, and you will want to compute the cosines of the angles that these vectors make with $\mathbf{a}$. If tomorrow you want to use the fewest number of operations (additions, multiplications, etc.), what computation(s) should you do today and why? How many operations will you save? [Justify your answer with a formula on the formula sheet.]

Answer: Today you can normalize a via $\mathbf{u} = \mathbf{a}/|\mathbf{a}|$, which costs 3 divisions (either way you have to compute $|\mathbf{a}|$). Then the formula

$$\cos\theta = \frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|\,|\mathbf{b}|}$$

is replaced with

$$\cos\theta = \frac{\mathbf{b}\cdot\mathbf{u}}{|\mathbf{b}|}$$

which tomorrow saves you one division (since $|\mathbf{u}| = 1$) for each of $1,000$ projection computations. Hence the total savings is $1,000$ minus the three divisions to find u, for a total savings of 997. Grading remark: one point for stating that

Continued on page 4

you normalize a today to simplify the computation tomorrow, two points for the correct operation count.

[3] **11.** Today you are given the constants a, b, c, d in the equation of a plane

$$ax + by + cz + d = 0.$$

Tomorrow you will be given 1,000 vectors, and you will want to compute the distances of these vectors to the plane. If tomorrow you want to use the fewest number of operations (additions, multiplications, etc.), what computation(s) should you do today and why? How many operations will you save? [Justify your answer with a formula on the formula sheet.]

Answer: Today you can normalize the equation $ax + by + cz + d = 0$ by dividing it by $|\langle a, b, c\rangle| = \sqrt{a^2 + b^2 + c^2}$; since $|\langle a, b, c\rangle|$ has to be computed either way, the cost of finding a', b', c', d' is 4 operations. This gives an equivalent equation $a'x + b'y + c'z + d' = 0$ but now $|\langle a', b', c'\rangle| = 1$; this means in the distance formula (on the sheet), which would have involved $|\langle a, b, c\rangle|$, now involves $|\langle a', b', c'\rangle|$. So the formula

$$|d - d_2|/|\langle a, b, c\rangle|$$

becomes

$$|d' - d_2|/|\langle a', b', c'\rangle| = |d' - d_2|,$$

and tomorrow we do not have to divide by the magnitude; the computation of d_2 has the same cost. This saves us one division for each distance computation (the rest of the computation is the same), for a total savings of 1,000 operations minus the cost of computing a', b', c', d' (which is 4 divisions); hence we save 996 operations in total. Grading remark: one point for stating that you normalize $ax + by + cz + d = 0$ today to simplify the computation tomorrow, two points for the correct operation count.

[3] **12.** You are given 1,000,000 3-dimensional vectors, and you want to determine which vectors are parallel (or nearly parallel). Which of the four methods discussed on the homework would you use to determine this using the fewest number of operations (additions, multiplications, etc.). How many operations will you save compared to the other methods? [Justify your answer.]

Answer: Finding unit vectors requires 9 operations (six to find the norm of a vector—one square root, two additions, and three multiplications—and three divisions) per vector, for 9,000,000 operations The other three methods require some number of operations (the scalar multiple method requires three divisions, the other two methods are more expensive) which much be done on all pairs of vectors, and there are roughly $10^{12}/2$ such pairs; so any of the other three methods require much more operations (at least $3/2$ times 10^{12}). Grading remark: one point for stating that finding unit vectors is the only

Continued on page 5

method that works on individual vectors rather than pairs, two points for the correct operation count.

[6] **13.** Find the intersection of the planes $x + y = 3$ and $x + z = 4$; notice that $(1, 2, 3)$ lies on both planes.

Answer: The direction of the line is the cross product of the normals to the planes, i.e.,

$$\langle 1, 1, 0\rangle \times \langle 1, 0, 1\rangle = \langle 1, -1, -1\rangle.$$

Since $(1, 2, 3)$ **lies on this line, the intersection is described by the line (in parametric form)** $\langle 1, 2, 3\rangle + \langle 1, -1, -1\rangle t$ **.**

[6] **14.** Find the intersection of the planes $2x + y = 3$ and $3y + z = 6$; notice that $(1, 1, 3)$ lies on both planes.

Answer: The direction of the line is the cross product of the normals to the planes, i.e.,

$$\langle 2, 1, 0\rangle \times \langle 0, 3, 1\rangle = \langle 1, -2, 6\rangle.$$

Since $(1, 1, 3)$ **lies on this line, the intersection is described by the line (in parametric form)** $\langle 1, 1, 3\rangle + \langle 1, -2, 6\rangle t$ **.**

[6] **15.** Find the intersection of the planes $x + 4y = 5$ and $y + z = 5$; notice that $(1, 1, 4)$ lies on both planes.

Answer: The direction of the line is the cross product of the normals to the planes, i.e.,

$$\langle 1, 4, 0\rangle \times \langle 0, 1, 1\rangle = \langle 4, -1, 1\rangle.$$

Since $(1, 1, 4)$ **lies on this line, the intersection is described by the line (in parametric form)** $\langle 1, 1, 4\rangle + \langle 4, -1, 1\rangle t$ **.**

[6] **16.** Find the intersection of the planes $x + y + z = 5$ and $y + 2z = 5$; notice that $(1, 3, 1)$ lies on both planes.

Answer: The direction of the line is the cross product of the normals to the planes, i.e.,

$$\langle 1, 1, 1\rangle \times \langle 0, 1, 2\rangle = \langle 1, -2, 1\rangle.$$

Since $(1, 3, 1)$ **lies on this line, the intersection is described by the line (in parametric form)** $\langle 1, 3, 1\rangle + \langle 1, -2, 1\rangle t$ **.**

[6] **17.** Find the plane through the three points $A(2, 1, 2)$, $B(1, 2, 3)$, and $C(0, 1, 2)$ using the cross product.

Answer: The vector from A **to** B **is** $(1, 2, 3) - (2, 1, 2) = \langle -1, 1, 1\rangle$**, and from** A **to** C **is** $(0, 1, 2) - (2, 1, 2) = \langle -2, 0, 0\rangle$**. So the normal to the plane is**

$$\langle -1, 1, 1\rangle \times \langle -2, 0, 0\rangle = \langle 0, -2, 2\rangle.$$

Continued on page 6

Hence the plane is described by $\langle 0,-2,2\rangle \cdot \langle x,y,z\rangle$ is constant, and this constant can be found by plugging in $A(2,1,2)$ for the x,y,z values, giving $\langle 0,-2,2\rangle \cdot \langle 2,1,2\rangle = 2$ (you could also plug in B or C, which would also give the constant of 2). Hence the equation is

$$-2y + 2z = 2$$

(or, more simply, $z - y = 1$).

[6] **18.** Find the distance from the point $A(1,2,3)$ to the plane $x + 2y - 2z = 5$.

Answer: A lies on the plane $x + 2y - 2z = 1 + 2 \cdot 2 - 2 \cdot 3 = -1$, so A's distance to $x + 2y - 2z = 5$ is the distance between the two planes $x + 2y - 2z - 5 = 0$ and $x + 2y - 2z + 1 = 0$ which (by the formula sheet) is

$$|-5-1|/\sqrt{1^2 + 2^2 + (-2)^2} = 6/3 = 2.$$

[6] **19.** Find the angle between the two planes $2x + 3y + 4z = 5$ and $x - 2y + z = 7$.

Answer: The two normals have a dot product of $\langle 2,3,4\rangle \cdot \langle 1,-2,1\rangle = 0$, so they are orthogonal, i.e., they make an angle of $90°$ (or $\pi/2$ radians); the planes make this angle as well.

[6] **20.** Find the angle between the line $\langle 2,3,1\rangle + t\langle 1,0,2\rangle$ and the plane $-2x - 4z = 5$.

Answer: The direction of the line is $\langle 1,0,2\rangle$, and the normal to the plane is $\langle -2,0,-4\rangle$; since these lines are parallel, the line makes an angle of $90°$ (i.e., $\pi/2$ radians) with the plane.

[6] **21.** Find the distance between the lines $\langle 1,1,0\rangle + t\langle 1,0,1\rangle$ and $\langle 3,0,9\rangle + t\langle 0,2,1\rangle$.

Answer: The direction perpendicular to the directions of the two lines is

$$\langle 1,0,1\rangle \times \langle 0,2,1\rangle = \langle -2,-1,2\rangle.$$

It follows that these two lines are not parallel. The first line lies on the plane

$$-2x - y + 2z = -2(1) - (1) + 2(0) = -3,$$

and the second line on

$$-2x - y + 2z = -2(3) - (0) + 2(9) = 12.$$

Hence the distance between the two lines is the distance between the two parallel planes $-2x - y + 2z + 3 = 0$ and $-2x - y + 2z - 12 = 0$, namely (by the formula sheet)

$$|3 - (-12)|/\sqrt{(-2)^2 + (-1)^2 + 2^2} = 15/3 = 5.$$

Continued on page 7

[6] **22.** Find the distance between the lines $\langle 2,0,1\rangle + t\langle 2,2,1\rangle$ and $\langle 5,0,1\rangle + t\langle 4,4,2\rangle$.

Answer: Since the directions of the line—namely $\langle 2,2,1\rangle$ and $\langle 4,4,2\rangle$—are parallel, the lines are parallel; since the vector

$$\mathbf{b} = \langle 5,0,1\rangle - \langle 2,0,1\rangle = \langle 3,0,0\rangle$$

points from the first line to the second, the projection of b onto the direction of $\mathbf{a} = \langle 2,2,1\rangle$ is

$$\mathbf{p} = \frac{\mathbf{b}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a} = \frac{6}{9}\langle 2,2,1\rangle = \langle 4/3,4/3,2/3\rangle.$$

It follows that the distance between the lines is

$$|\mathbf{b}-\mathbf{p}| = |\langle 3,0,0\rangle - \langle 4/3,4/3,2/3\rangle| = |\langle 5/3,-4/3,-2/3\rangle| = \sqrt{45}\,/3 = \sqrt{5}.$$

[6] **23.** Find the distance between the line $\langle 0,2,1\rangle + t\langle 4,3,1\rangle$ and the plane $x-2y+2z=34$.

Answer: Since the direction of the line (namely $\langle 4,3,1\rangle$) is orthogonoal to the normal to the plane (namely $\langle 1,-2,2\rangle$), the line is contained in a plane $x-2y+2z+d=0$, where

$$d = -x+2y-2z = -(0)+2(2)-2(1) = 2.$$

Hence the distance we seek is the same as the distance from $x-2y+2z-34=0$ and $x-2y+2z+2=0$. Using the formula sheet, we see that this distance is

$$|-34-2|/\sqrt{1^2+(-2)^2+2^2} = 36/3 = 12.$$

[6] **24.** Find the area of the parallelogram in the plane $z=2x+y-1$ defined by $0\le x\le 1$ and $0\le y\le 1$.

Answer: The vertex of the parallelogram with $x=0$ and $y=0$ has $z = 2\cdot 0+0-1=-1$, i.e., is the point $(0,0,-1)$; the vertex of the parallelogram with $x=1$ and $y=0$ is similarly $(1,0,1)$; the vertex of the parallelogram with $x=0$ and $y=1$ is similarly $(0,1,0)$. It follows that the sides of the parallelogram point in the direction

$$\langle 1,0,1\rangle - \langle 0,0,-1\rangle = \langle 1,0,2\rangle$$

and

$$\langle 0,1,0\rangle - \langle 0,0,-1\rangle = \langle 0,1,1\rangle\ .$$

It follows that the area of the parallelogram is

$$|\langle 1,0,2\rangle \times \langle 0,1,1\rangle| = |\langle -2,-1,1\rangle| = \sqrt{6}.$$

Continued on page 8

Continued on page 9

The End

Be sure that this examination has 6 pages including this cover

The University of British Columbia

Midterm Examinations - October 2015

Mathematics 200-102

Closed book examination

Time: 45 minutes

Name ____________________ **Signature** ____________________

Student Number ____________________ **Instructor's Name** ____________________

Section Number ____________________

1		1
2		1
3		1
4		1
5		2
6		2
7		2
8		2
9		3
10		3
11		3
12		3
13		6
14		6
15		6
16		6
17		6
18		6
19		6
20		6
21		6

Special Instructions:

Calculators, notes, or other aids may not be used. Answer questions on the exam. This exam is two-sided!

Rules governing examinations

1. Each candidate should be prepared to produce his library/AMS card upon request.

2. Read and observe the following rules:

No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.

Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.