WRITTEN HOMEWORK 9 (SOLUTIONS), MATH 200, FALL 2015

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This homework **may be modified from section to section!** Check your section's website for any modifications to this homework for your section.

Final 2012WT1, Problem 8

[Done via polar coordinates.]

The equation $x^2 + y^2 = x$ is the same as $(x^2 - x) + y^2 = 0$, which by completing the square (as in Section 12.1 of the textbook)

$$(x - 1/2)^2 + y^2 = x^2 - x + (1/4) + y^2 = 1/4,$$

which is the circle of radius 1/2 centred at $(1/2, 0)$. Hence the interior of this circle lies in the unit disc, and is described by the equation

$$(x - 1/2)^2 + y^2 \le 1/4, \quad \text{i.e.,} \quad x^2 + y^2 \le x;$$

in polar coordinates this amounts to

$$r^2 \le r\cos\theta\ .$$

This always holds at $r = 0$; for $r > 0$ this means that $r \le \cos\theta$, which is impossible for θ with $\cos\theta < 0$, while for θ with $\cos\theta \ge 0$ this region is desribed by

$$0 \le r \le \cos\theta.$$

Hence the crescent, which is the unit disc with the above circle interior taken away, descibed by

$$\cos\theta \le r \le 1, \quad \text{for } -\pi/2 \le \theta \le \pi/2, \text{ and}$$
$$0 \le r \le 1, \quad \text{for } \pi/2 \le \theta \le 3\pi/2.$$

It follows that the mass of the crescent with unit density is

$$\int_{\theta=-\pi/2}^{\theta=\pi/2} \int_{r=\cos\theta}^{r=1} r\,dr\,d\theta + \int_{\theta=\pi/2}^{\theta=3\pi/2} \int_{r=0}^{r=1} r\,dr\,d\theta$$
$$= \int_{\theta=-\pi/2}^{\theta=\pi/2} r^2/2 \Big|_{r=\cos\theta}^{r=1} d\theta + \int_{\theta=\pi/2}^{\theta=3\pi/2} r^2/2 \Big|_{r=0}^{r=1} d\theta$$
$$= \int_{\theta=-\pi/2}^{\theta=\pi/2} (1 - \cos^2\theta)/2\,d\theta + \int_{\theta=\pi/2}^{\theta=3\pi/2} (1/2)\,d\theta$$
$$= \int_{\theta=-\pi/2}^{\theta=\pi/2} \sin^2\theta(1/2)\,d\theta + \pi/2$$

which, using $\sin^2\theta = (1-\cos(2\theta))/2$ is

$$= \int_{\theta=-\pi/2}^{\theta=\pi/2} (1-\cos(2\theta))/4\, d\theta + \pi/2 = \pi/4 + \pi/2 = 3\pi/4.$$

We similarly compute the integral over the crescent of $x = r\cos\theta$ to be

$$\int_{\theta=-\pi/2}^{\theta=\pi/2}\int_{r=\cos\theta}^{r=1} r^2\cos\theta\, dr\, d\theta + \int_{\theta=\pi/2}^{\theta=3\pi/2}\int_{r=0}^{r=1} r^2\cos\theta\, dr\, d\theta$$

which, since $r^2\,dr$ integrates to $(1/3)r^3$, is

$$= \int_{\theta=-\pi/2}^{\theta=\pi/2} (1/3)(1-\cos^3\theta)\,\cos\theta\, d\theta + \int_{\theta=\pi/2}^{\theta=3\pi/2} (1/3)\cos\theta\, d\theta$$

$$= \int_{\theta=-\pi/2}^{\theta=\pi/2} -(1/3)\cos^4\theta\, d\theta + \int_{\theta=-\pi/2}^{\theta=\pi/2} (1/3)\cos\theta\, d\theta + \int_{\theta=\pi/2}^{\theta=3\pi/2} (1/3)\cos\theta\, d\theta$$

$$= \int_{\theta=-\pi/2}^{\theta=\pi/2} -(1/3)\cos^4\theta\, d\theta + \int_{\theta=-\pi/2}^{\theta=3\pi/2} (1/3)\cos\theta\, d\theta\ .$$

Since the integral of $\cos\theta$ is $\sin\theta$ we see that the second integral above is 0, which leaves the above equal to

$$\int_{\theta=-\pi/2}^{\theta=\pi/2} -(1/3)\cos^4\theta\, d\theta\ ;$$

by the formula on the sheet this integral equals $-\pi/8$. Hence, setting R to be the crescent interior, we have—given the constant density of 1—that

$$\overline{x} = \frac{\int\int_R x\, dA}{\int\int_R dA} = \frac{-\pi/8}{3\pi/4} = -1/6.$$

Problem 2

We have

$$\int\int_D x\, dA = \int_{x=x_0-r}^{x=x_0+r}\int_{y=y_0-\sqrt{r-(x-x_0)^2}}^{y=y_0+\sqrt{r-(x-x_0)^2}} x\, dy\, \mathrm{d}x$$

$$= \int_{x=x_0-r}^{x=x_0+r} 2\sqrt{r^2-(x-x_0)^2}\, x\, dx\ . \tag{1}$$

We evaluate this integral by substitution: recall that to integrate $(1-t^2)^{1/2}$ we use the substitution $t=\cos\phi$; similarly to integrate $(a^2-t^2)^{1/2}$ for any constant a we set $t = a\cos\phi$; the analogous substitution here is $x - x_0 = r\cos\theta$: the main justification for this is that

$$\sqrt{r^2-(x-x_0)^2} = \sqrt{r^2-r^2\cos^2\theta} = \sqrt{r^2\sin^2\theta} = r|\sin\theta|,$$

and

$$dx = r(-\sin\theta)\, d\theta\ ;$$

also note that $\theta = \pi$ at $x = x_0 - r$ and $\theta = 0$ at $x = x_0 + r$; furthermore $\sin\theta \ge 0$ for θ between π and 0, so $|\sin\theta| = \sin\theta$ in this range. So the above integral (Equation 1) becomes (remember that $x = x_0 + r\cos\theta$)

$$\int_{\theta=\pi}^{\theta=0} (r\,2\sin\theta)(x_0 + r\cos\theta)(-r\sin\theta\,d\theta) = \int_{\theta=0}^{\theta=\pi} (2r^2 x_0 \sin^2\theta + r^3\sin^2\theta\cos\theta)\,d\theta$$

(where the right-hand-side is the left-hand-side with the integration limits $\theta = 0$ and $\theta = \pi$ exchanged, getting rid of a minus sign in the process)

$$= \int_{\theta=0}^{\theta=\pi} 2r^2 x_0 \sin^2\theta\,d\theta + \int_{\theta=0}^{\theta=\pi} r^3\sin^2\theta\cos\theta\,d\theta\ .$$

The $\sin^2\theta\cos\theta$ integrates to $\sin^3\theta/3$, and to integrate $\sin^2\theta$ we use $\sin^2\theta = (1 - \cos(2\theta))/2$, whose integral is $\theta/2 - \sin(2\theta)/4$; hence the above integral becomes

$$\Big[2r^2 x_0\big(\theta/2 - \sin(2\theta)/4\big) + r^3(\sin^3\theta)/3\Big]\ \Big|_{\theta=0}^{\theta=\pi}$$

$$= \Big[2r^2 x_0(\pi/2) - 0 + r^3\,0\Big] - \Big[2r^2 x_0\,0 - 0 + r^3\,0\Big] = x_0\pi r^2\ .$$

Hence

$$\int\int_R x\,dA = x_0\pi r^2\ .$$

Intuitively we know that (by "symmetry") the centre of mass of a circle should be its centre, so $\overline{x}$, its average x-coordinate, should be x_0 times the area of the circle interior, πr^2.

Problem 3

Let C_1 be the interior of the unit circle, $x^2 + y^2 = 1$, and C_2 be the interior of the circle within $x^2 + y^2 = x$, so that the crescent, R, in the final exam problem above is just C_1 with C_2 removed.

We have

$$\int\int_R dA = \mathrm{area}(C_1) - \mathrm{area}(C_2) = \pi r^2 - \pi(r/2)^2 = \pi r^2(3/4).$$

Similarly, using Problem 2, we have

$$\int\int_R x\,dA = \int\int_{C_1} x\,dA - \int\int_{C_2} x\,dA$$

$$= 0\cdot\mathrm{area}(C_1) - (1/2)\cdot\mathrm{area}(C_2) = -(1/2)\pi(1/2)^2 = -\pi/8,$$

and so the centre of mass of R with constant density 1 is

$$\overline{x} = \frac{\int\int_R x\,dA}{\int\int_R dA} = \frac{-\pi/8}{3\pi/4} = -1/6.$$

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