WRITTEN HOMEWORK 7 (SOLUTIONS), MATH 200, FALL 2015

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This homework **may be modified from section to section!** Check your section's website for any modifications to this homework for your section.

Final 2013WT1, Problem 3(a)

We have

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle 2(x-2), 2(y-1), 2z \rangle$$

and for the constraint $g(x,y,z) = 1$ with $g = x^2 + y^2 + z^2$

$$\nabla g = \langle g_x, g_y, g_z \rangle = \langle 2x, 2y, 2z \rangle.$$

Solving $\nabla f = \lambda \nabla g$ gives

$$x - 2 = \lambda x, \quad y - 1 = \lambda y, \quad z = \lambda z.$$

Perhaps the simplest place to start is $z = \lambda z$: this means that either $\lambda = 1$ or $z = 0$.

(1) Case $\lambda = 1$: so z can be anything (so far), but $x-2 = x$ which is impossible.
(2) Case $z = 0$: so λ can be anything so far, but based on $y - 1 = \lambda y$ and $x - 2 = \lambda x$ we can eliminate λ: we have

$$\lambda = (y-1)/y = (x-2)/x,$$

unless either y or x is zero. So we get some subcases:

(a) y and x not zero: $(y-1)/y = (x-2)/x$, so cross multiplying gives $y(x-2) = x(y-1)$, and hence $x = 2y$. Since $z = 0$ already, and now $x = 2y$, we get

$$1 = x^2 + y^2 + z^2 = (2y)^2 + y^2 + 0^2 = 5y^2$$

so $y = \pm 1/\sqrt{5}$ and $x = 2y$. So we check

$$f(2/\sqrt{5}, 1/\sqrt{5}, 0) = 6 - 10/\sqrt{5}, \quad f(-2/\sqrt{5}, -1/\sqrt{5}, 0) = 6 + 10/\sqrt{5}.$$

(b) $y = 0$: so also $z = 0$, so $x^2 + y^2 + z^2 = 1$ implies that $x = \pm 1$, and we find

$$f(1,0,0) = 2, \quad f(-1,0,0) = 10$$

(the value of λ not particularly needed).

(c) $x = 0$: simimilarly $y = \pm 1$, and we find

$$f(0,1,0) = 4, \quad f(0,-1,0) = 8.$$

So the min of f on $g = 1$ is the smallest and largest from among the values

$$6 \pm 10/\sqrt{5}, 2, 10, 4, 8,$$

namely $6 - 10/\sqrt{5}$.

Final 2013WT1, Problem 4(b)

With notation as above, we have

$$f(x, y, z) = (x^2 + y^2 + z^2 + 5) - 4x - 2y.$$

Since the constraint reads $z^2 = 1 - x^2 + y^2$, we can write that

$$x^2 + y^2 + z^2 = 1 \quad \text{implies} \quad f(x, y, z) = 6 - 4x - 2y.$$

But this expression $6 - 4x - 2y$ does not involve z; the only constraint on x and y is that for some z we have $x^2 + y^2 + z^2 = 1$, and this means that $x^2 + y^2 \leq 1$. So Problem 3(a) is equivalent to minimizing $h(x, y) = 6 - 4x - 2y$ on the region $x^2 + y^2 = 1$.

Final 2012WT1, Problem 6(i)

Here we have

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle 2(x - 2), 2(y + 2), 2(z - 4) \rangle$$

and for the constraint $g(x, y, z) = 6$ with $g = x^2 + y^2 + z^2$

$$\nabla g = \langle g_x, g_y, g_z \rangle = \langle 2x, 2y, 2z \rangle.$$

Solving $\nabla f = \lambda \nabla g$ gives

$$x - 2 = \lambda x, \quad y + 2 = \lambda y, \quad z - 4 = \lambda z.$$

This means that

$$\lambda = \frac{x - 2}{x} = \frac{y + 2}{y} = \frac{z - 4}{z}$$

provided that x, y, z are all non-zero. So we get the following two cases: x, y, z are all non-zero, or at least one of x, y, z is zero.

(1) In case x, y, z are all non-zero: then

$$\frac{x - 2}{x} = \frac{y + 2}{y} = \frac{z - 4}{z},$$

an in particular

$$\frac{x - 2}{x} = \frac{y + 2}{y} \quad \text{so} \quad (x - 2)y = (y + 2)x$$

so $-2y = 2x$ or $x = -y$; also

$$\frac{y + 2}{y} = \frac{z - 4}{z} \quad \text{so} \quad (y + 2)z = (z - 4)y$$

so $2z = -4y$ so $z = -2y$. So we get $x = -y$ and $z = -2y$ and we solve

$$6 = x^2 + y^2 + z^2 = (-y)^2 + y^2 + (-2y)^2 = 6y^2,$$

so $y = \pm 1$, $\langle x, y, z \rangle = \pm \langle -1, 1, -2 \rangle$ and we check

$$f(-1, 1, -2) = (-1 - 2)^2 + (1 + 2)^2 + (-2 - 4)^2 = 9 + 9 + 36 = 54,$$

and

$$f(1,-1,2) = (1-2)^2 + (-1+2)^2 + (2-4)^2 = 1+1+4 = 6.$$

(2) In case at least one of x, y, z is zero: lets say $x = 0$. Then we have

$$x - 2 = \lambda x = 0,$$

so $x-2=0$, which is impossible. Similarly $y=0$ and $z=0$ are impossible.

Hence the min of f subject to $g = 6$ is 6, and its max is 54.

Final 2012WT1, Problem 6(ii)

The point that is farthest from $(2,-2,4)$ is just the point that maximizes

$$(x-2)^2 + (y+2)^2 + (z-4)^2.$$

But this just happens to be the function f in Problem 6(i). So we already know that this function is maximized at $\langle -1, 1, -2 \rangle$.

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