WRITTEN HOMEWORK 6 (SOLUTIONS), MATH 200, FALL 2015

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This homework **may be modified from section to section!** Check your section's website for any modifications to this homework for your section.

Final 2013WT2, Problem 3

We have

$$\nabla f = (f_x, f_y) = (6kxy - 6x, 3kx^2 + 3y^2 - 6y).$$

Solving for $\nabla f = (0,0)$, we have that $f_x = 0$ implies that $6x(ky-1) = 0$, which gives two cases:

(1) Case $x = 0$: then $f_y = 3y^2 - 6y$ so $f_y = 0$ implies that $3y(y-2) = 0$, i.e., either $y = 0$ or $y = 2$. So this case yields the critical points $(0,0)$ and $(0,2)$. For the second derivative test we have

(1)

$$f_{xx} = (6kxy - 6x)_x = 6ky - 6, \quad f_{xy} = (6kxy - 6x)_y = 6kx, \quad f_{yy} = (3kx^2 + 3y^2 - 6y)_y = 6y - 6,$$

and therefore

$$D = f_{xx}f_{yy} - (f_{xy})^2 = (6ky-6)(6y-6) - (6kx)^2 = 36[(ky-1)(y-1) - k^2x^2].$$

Hence

(a) At $(0,0)$ we have $f_{xx} = -6$ and $D = 36$, so $(0,0)$ is a local maximum; and

(b) at $(0,2)$ we have $f_{xx} = 12k - 6$ and $D = 36(2k-1)$, so when $k = 1/2$ then $f_{xx} = D = 0$ so the critical point is indeterminate; when $k > 1/2$ then $f_{xx} > 0$ and $D > 0$ so the point is a local maximum; and when $k < 1/2$ then $D < 0$ so the point is a saddle.

(2) Case $ky = 1$, i.e., $y = 1/k$: here we have

$$f_y = 3kx^2 + 3y^2 - 6y = 3kx^2 + 3/k^2 - 6/k,$$

so $f_y = 0$ implies that $3kx^2 = 6/k - 3/k^2$ so $x^2 = 2/k^2 - 1/k^3$ so

$$x = \pm\sqrt{2/k^2 - 1/k^3} = \pm\sqrt{2 - 1/k}\,/k.$$

So if $k = 1/2$ we have $x = 0$, $y = 2$, which was covered above. If $k < 1/2$, then there are no real values of x. If $k > 1/2$, then there are the two x values above to check. Since

$$f_{xx} = (6kxy - 6x)_x = 6ky - 6, \quad f_{xy} = (6kxy - 6x)_y = 6kx, \quad f_{yy} = (3kx^2 + 3y^2 - 6y)_y = 6y - 6,$$

for $ky = 1$ we have $f_{xx} = 0$. So

$$D = f_{xx}f_{yy} - (f_{xy})^2 = -36k^2x^2 < 0,$$

so $y = 1/k$ and $x = \pm\sqrt{2-1/k}\,/k$ are saddles.

Final 2012WT1, Problem 3

By the chain rule we have

$$G_t = \frac{\partial G}{\partial t} = \frac{\partial}{\partial t}F(\gamma+s, \gamma-s, At)$$

$$= F_x(\gamma+s)_t + F_y(\gamma-s)_t + F_z(At)_t = F_x\cdot 0 + F_y\cdot 0 + F_z A = A\,F_z = A\,F_z(\gamma+s, \gamma-s, At).$$

(It is important to remember that F, F_z, etc. are being evaluated at the point $(x, y, z) = (\gamma+s, \gamma-s, At)$; it is a bit cumbersome to put this everywhere.) Similarly,

$$G_\gamma = F_x(\gamma+s)_\gamma + F_y(\gamma-s)_\gamma + F_z(At)_\gamma = F_x + F_y = F_x(\gamma+s, \gamma-s, At) + F_y(\gamma+s, \gamma-s, At).$$

Similarly

$$G_{\gamma\gamma} = (G_\gamma)_\gamma = [F_x(\gamma+s, \gamma-s, At)]_\gamma + [F_y(\gamma+s, \gamma-s, At)]_\gamma = [F_{xx}+F_{xy}] + [F_{yx}+F_{yy}]$$

Similarly

$$G_s = F_x - F_y, \quad G_{ss} = F_{xx} - F_{xy} - F_{yx} + F_{yy}.$$

Hence

$$G_{\gamma\gamma} + G_{ss} = 2F_{xx} + 2F_{yy} = 2F_z,$$

by the equation for F. Since $G_t = A\,F_z$, we have $G_t = G_{\gamma\gamma} + G_{ss}$ iff $2\,F_z = A\,F_z$, which holds if $A = 2$.

Problem 34, Section 14.7

We need to find the min/max of $f(x, y) = xy^2$ in the region D described by the inequalities $x \geq 0$, $y \geq 0$, and $x^2 + y^2 \leq 3$.

We have

$$\nabla f = (f_x, f_y) = (y^2, 2xy).$$

So if $\nabla f = (0, 0)$ we have $f_x = 0$ and therefore $y = 0$. But y is never 0 in the interior of D (i.e., for $x > 0$, $y > 0$, and $x^2 + y^2 < 3$, where y must be positive). Hence ∇f is never zero in the interior of D, and it suffices to check the values of f on the boundary of D.

On the boundary where $x = 0$ or $y = 0$ we have that $f = xy^2 = 0$. On the boundary where $x^2 + y^2 = 3$ (and both x and y are non-negative) we have

$$f = xy^2 = x(3 - x^2),$$

and x ranges from 0 to $\sqrt{3}$. So aside from the values where $f = 0$, the only other possible min/max values of f occur for the function $g(x) = x(3 - x^2)$ with $x \in (0, \sqrt{3})$: since

$$g'(x) = (3x - x^3)' = 3 - 3x^2,$$

we have $g'(x) = 0$ for $x = \pm 1$; since we only are considering $x \in (0, \sqrt{3})$, $g' = 0$ there only for $x = 1$; furthermore $g(1) = 1 \cdot (3 - 1^2) = 2$.

So the only possibly min/max values of $f(x, y)$ in D are the values 0 and 2; hence 0 is the minimum value of f, and 2 is the maximum value of f.

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