WRITTEN HOMEWORK 2, MATH 200, FALL 2015

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This homework **may be modified from section to section!** Check your section's website for any modifications to this homework for your section.

Problem 1

We now have four ways to determine if two vectors are parallel: the two on Written Homework 1, and the following two methods:

(1) $\mathbf{a}$ and $\mathbf{b}$ are parallel iff

$$\mathbf{a}\cdot\mathbf{b} = \pm\,|\mathbf{a}|\,|\mathbf{b}|,$$

and

(2) $\mathbf{a}$ and $\mathbf{b}$ are parallel iff

$$\mathbf{a}\times\mathbf{b} = \mathbf{0}$$

Use both of the above two methods to determine

(1) if $\mathbf{a} = \langle 1,3,2\rangle$ is parallel to $\mathbf{b} = \langle 5,15,10\rangle$;
(2) if $\mathbf{a} = \langle 1,3,2\rangle$ is parallel to $\mathbf{b} = \langle 5,15,12\rangle$; and
(3) if $\mathbf{a} = \langle 1,3,2\rangle$ is parallel to $\mathbf{b} = \langle -6,-18,-12\rangle$.

Problem 1, Solutions:

(1) $\mathbf{a} = \langle 1,3,2\rangle$ and $\mathbf{b} = \langle 5,15,10\rangle$:

$$\mathbf{a}\cdot\mathbf{b} = 1\cdot 5 + 3\cdot 15 + 2\cdot 10 = 5 + 45 + 20 = 70$$

while

$$|\mathbf{a}|\,|\mathbf{b}| = \sqrt{1^2+3^2+2^2}\sqrt{5^2+15^2+10^2} = \sqrt{14}\sqrt{350} = \sqrt{4900} = 70,$$

so $\mathbf{a}$ and $\mathbf{b}$ are parallel by the first method above.

$$\mathbf{a}\times\mathbf{b} = \langle 3\cdot 10 - 2\cdot 15, -(1\cdot 10 - 2\cdot 5), 1\cdot 15 - 3\cdot 5\rangle = \langle 0,0,0\rangle,$$

so $\mathbf{a}$ and $\mathbf{b}$ are parallel by the second method above.

(2) $\mathbf{a} = \langle 1,3,2\rangle$ and $\mathbf{b} = \langle 5,15,12\rangle$:

$$\mathbf{a}\cdot\mathbf{b} = 5 + 45 + 24 = 74$$

while

$$|\mathbf{a}|\,|\mathbf{b}| = \sqrt{1^2+3^2+2^2}\sqrt{5^2+15^2+12^2} = \sqrt{14}\sqrt{394} = \sqrt{5516} = 74.26977\ldots \neq \pm 74$$

so **a** and **b** are not parallel by the first method above.

$$\mathbf{a}\times\mathbf{b} = \langle 3\cdot 12 - 2\cdot 15, -(1\cdot 12 - 2\cdot 5), 1\cdot 15 - 3\cdot 5\rangle = \langle 6,-6,0\rangle,$$

so **a** and **b** are not parallel by the second method above.

(3) $\mathbf{a} = \langle 1,3,2\rangle$ and $\mathbf{b} = \langle -6,-18,-12\rangle$:

$$\mathbf{a}\cdot\mathbf{b} = -6-54-24 = -84$$

while

$$|\mathbf{a}|\,|\mathbf{b}| = \sqrt{14}504 = 84 = -\mathbf{a}\cdot\mathbf{b},$$

so **a** and **b** are parallel by the first method above.

$$\mathbf{a}\times\mathbf{b} = \langle 3(-12)-2(-18), -(1(-12)-2(-6)), 1(-18)-3(-12)\rangle = \langle 0,0,0\rangle,$$

so **a** and **b** are parallel by the second method above.

Problem 2

We shall use the fact (Exercise 53, Section 12.3) that in $\mathbb{R}^2$ (i.e., the plane), the distance from a point $P_1(x_1, y_1)$ to the line $ax + by + c = 0$ in the (x, y)-plane is

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}. \tag{1}$$

Say that the equation $ax+by+c = 0$ (as in Problem 2) is *normalized* if $a^2+b^2 = 1$.

(1) How does Equation 1 simplify if we know that $a^2 + b^2 = 1$, i.e., if the equation is normalized? Explain.

(2) The equation of the line in the plane

$$3x + 4y - 2 = 0$$

can be divided by $|\langle 3, 4\rangle| = 5$ to get an equivalent equation

$$(0.6)x + (0.8)y - (0.4) = 0$$

which is normalized. Using a similar idea, write an equation that is equivalent to

$$5x + 12y - 26 = 0$$

that is normalized. Similarly for the equation

$$8x - 6y + 15 = 0.$$

(3) Recall the "number of operations" as explained on Problem 4 of Written Homework 1, and in their (soon to be published) solutions. If you are given an equation of one line, $ax + by + c = 0$ and 1000 points whose distance from the line you wish to compute, what is the advantage—in terms of computation speed (i.e., numbers of operations) in first normalizing the equation $ax + by + c = 0$? Explain.

Problem 2, Solutions:

(1) Then the distance from $P_1(x_1, y_1)$ to the line becomes

$$|ax_1 + by_1 + c|.$$

(2) Dividing $5x + 12y - 26 = 0$ by $|\langle 5, 12\rangle| = 13$ gives the normalized equation

$$(5/13)x + (12/13)y - 2 = 0$$

or roughly

$$(.3846153846)x + (.9230769231)y - 2 = 0$$

(either form is acceptable). Dividing $8x - 6y + 15 = 0$ by $|\langle 8, -6\rangle| = 10$ yields the normalized equation

$$(0.8)x + (-0.6)y + 1.5 = 0.$$

(3) If you normalize first you save yourself having to divide by $\sqrt{a^2 + b^2}$ in the equation at the beginning of this exercise. Either way you will want to compute $\sqrt{a^2 + b^2}$; by normalizing first you divide a and b by $\sqrt{a^2 + b^2}$ which costs you 2 divisions; if you don't normalize, you need to divide by $\sqrt{a^2 + b^2}$ in each distance computation, for a total of 1000 divisions. Aside from this, the computations—and therefore operation counts—are the same. So by finding the normalized equation you save yourself $1000 - 2 = 998$ operations.

Problem 3

Consider the formula:

$$\mathrm{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a}$$

(1) How does this formula simplify if **a** happens to be a unit vector?
(2) Given a single vector, **a**, and 1000 vectors whose projection onto **a** we wish to compute, can we speed up this computation (in terms of number of operations, as in Problem 2) by first computing $\mathbf{u} = \mathbf{a}/|\mathbf{a}|$? Explain.

Problem 3, Solutions:

(1) In this case $\mathbf{a}\cdot\mathbf{a} = 1$ or, equivalently, $|\mathbf{a}| = 1$, and the formula becomes

$$\mathrm{proj}_{\mathbf{a}}\mathbf{b} = (\mathbf{a}\cdot\mathbf{b})\mathbf{a}$$

(2) Let us first assume that **a** is 3-dimensional. Let us focus on how the two computations differ.

If we do not compute u, then we need to compute $\mathbf{a}\cdot\mathbf{a}$, and for each of 1000 vectors **b** we need to compute $\mathbf{a}\cdot\mathbf{b}$, divide this by $\mathbf{a}\cdot\mathbf{a}$, and multiply by each of the three components of **a**.

If we first compute $\mathbf{u} = \mathbf{a}/|\mathbf{a}|$, then we must compute $|\mathbf{a}|$, which requires a single square root one we compute $\mathbf{a}\cdot\mathbf{a}$, and then we divide the three components of **a** by $|\mathbf{a}|$. Then for each of 1000 **b** we compute the projection using **u** instead of **a**, which means that we save ourselves one division for each projection computations. So computing **u** costs ourselves four more operations, and saves 1000 operations. In total we save 996 operations by first normalizing **a**, i.e., replacing it with **u**.

[Since the problem does not state that our vectors are necessarily 3-dimensional, you could have answered this question for other dimensions. (I imagine most people didn't do this. . .) If you did this in n-dimensions, then computing **u** would cost you $n+1$ operations (the square root computation, and then dividing each of **a**'s components by n), and using **u** instead of **a** saves you 1000 operations (one division for each of 1000 vectors **b**). So interestingly, if n is much larger than 1000, then actually working with **u** is more expensive ($n+1$ versus 1000).]

Problems 4 and 5, Solutions

See the PDF files attached below.

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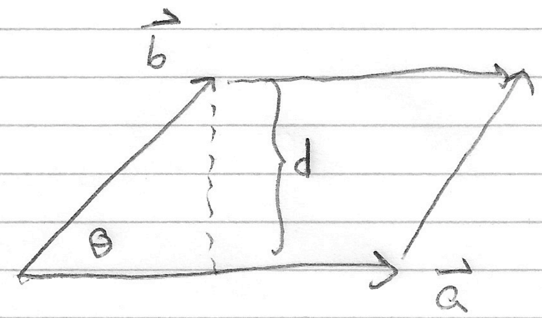
Two methods:



d = distance = height of parallelogram with $\vec{a}$ as base

① Area(Parallelogram) = base · height

$$|\vec{a} \times \vec{b}| = |\vec{a}| \cdot d$$

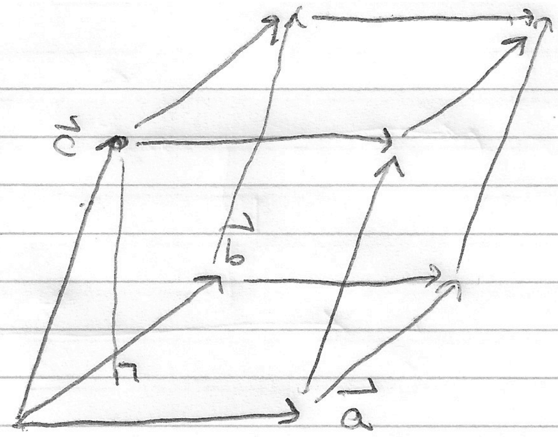
so $$d = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|}$$ (one method)

② $$d = |\vec{b}|\,|\sin\theta| = \frac{|\vec{a}|\,|\vec{b}|\,|\sin\theta|}{|\vec{a}|} = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|}$$

(second method)

There are probably other methods ...

Two methods:



d = distance = height of parallelopiped with base being the parallelogram given by $\vec{a}, \vec{b}, \vec{c}$

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(1) Volume(Parallelopiped) = (area of base) $\cdot$ (height)

$$\left| \vec{a} \cdot (\vec{b} \times \vec{c}) \right| = \left| \vec{a} \times \vec{b} \right| \cdot d$$

so $$d = \frac{\left| \vec{a} \cdot (\vec{b} \times \vec{c}) \right|}{\left| \vec{a} \times \vec{b} \right|}$$

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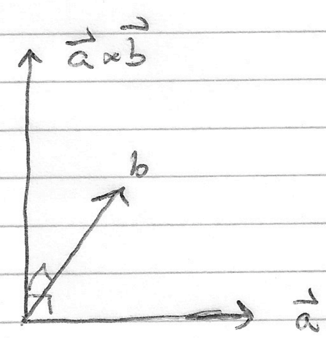
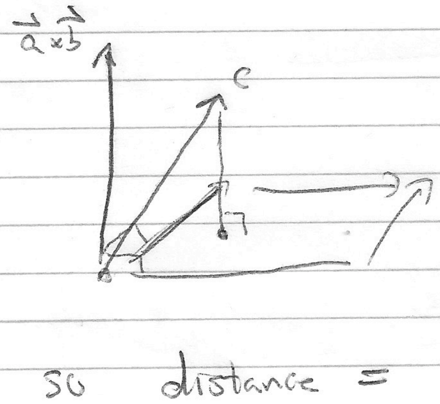
(2)

$\vec{a} \times \vec{b}$ is orthog to $\vec{a}, \vec{b}$





so distance = magnitude of $\text{proj}_{\vec{a} \times \vec{b}}(\vec{c})$

$$= \left| \frac{(\vec{a} \times \vec{b}) \cdot \vec{c}}{(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b})} (\vec{a} \times \vec{b}) \right|$$

$$= \frac{\left| (\vec{a} \times \vec{b}) \cdot \vec{c} \right|}{\left| \vec{a} \times \vec{b} \right|} = \frac{\left| \vec{a} \cdot (\vec{b} \times \vec{c}) \right|}{\left| \vec{a} \times \vec{b} \right|}$$

true by "triple product (and determinant) identities"