SOLUTIONS TO WRITTEN HOMEWORK 1, MATH 200, FALL 2015

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This homework **may be modified from section to section!** Check your section's website for any modifications to this homework for your section.

Recall (Section 12.2) that nonzero vectors two vectors are *parallel* if they are scalar multiples of each other.

Problem 1

One way to determine if two vectors are parallel is to see if one is a scalar multiple of another. For example, to see if $\mathbf{a} = \langle 1, 3, 2\rangle$ is parallel to $\mathbf{b} = \langle 2, 6, 5\rangle$, one writes the equation

$$\mathbf{b} = c\,\mathbf{a}\ ,$$

for a scalar (i.e.,real number) c and sees if there is a solution. In this case the equation amounts to

$$\langle 2, 6, 5\rangle = c\,\langle 1, 3, 2\rangle = \langle c, 3c, 2c\rangle,$$

which means that we are looking for a c such that

$$2 = c, \quad 6 = 3c, \quad 5 = 2c$$

which has no solution. On the other hand if $\mathbf{d} = \langle 10, 30, 20\rangle$, then $\mathbf{d}$ and $\mathbf{a}$ are parallel, since the equation

$$\langle 10, 30, 20 = c\,\langle 1, 3, 2\rangle = \langle c, 3c, 2c\rangle$$

has a solution, namely $c = 10$.

Use the above method to determine

(1) if $\mathbf{a} = \langle 1, 3, 2\rangle$ is parallel to $\mathbf{b} = \langle 5, 15, 10\rangle$;
(2) if $\mathbf{a} = \langle 1, 3, 2\rangle$ is parallel to $\mathbf{b} = \langle 5, 15, 12\rangle$; and
(3) if $\mathbf{a} = \langle 1, 3, 2\rangle$ is parallel to $\mathbf{b} = \langle -6, -18, -12\rangle$.

Problem 1: Solutions

(1) For $\mathbf{a} = \langle 1, 3, 2\rangle$ and $\mathbf{b} = \langle 5, 15, 10\rangle$, we solve
$$\langle 1, 3, 2\rangle = c\langle 5, 15, 10\rangle,$$
which means
$$1 = 5c, \quad 3 = 15c, \quad 2 = 10c\ ;$$
so $c = 1/5$ satisfies all these equations, and hence the vectors in question are parallel.

(2) For $\mathbf{a} = \langle 1, 3, 2\rangle$ and $\mathbf{b} = \langle 5, 15, 12\rangle$ we similarly solve
$$1 = 5c, \quad 3 = 15c, \quad 2 = 12c\ ;$$
the first two equations require $c = 1/5$ while the third requires $c = 1/6$, so there is no common solution c; hence the vectors in question are not parallel.

(3) For $\mathbf{a} = \langle 1, 3, 2\rangle$ and $\mathbf{b} = \langle -6, -18, -12\rangle$ we solve
$$1 = -6c, \quad 3 = -18c, \quad 2 = -12c,$$
which have a common solution $c = -1/6$; hence the vectors in question are parallel.

Problem 2

Recall (Section 12.2) that to each nonzero vector $\mathbf{a}$, the vector $\mathbf{u} = \mathbf{a}/|\mathbf{a}|$ is the unique unit vector (i.e., vector of length 1) whose direction is the same as $\mathbf{a}$.

For each vector below, find the unique unit vector that points in the same direction:

(1) $\mathbf{a} = \langle 3, 0, -4\rangle$;
(2) $\mathbf{b} = \langle 6, 0, 8\rangle$;
(3) $\mathbf{c} = \langle -3, 0, 4\rangle$;
(4) $\mathbf{d} = \langle 3, 0, 4\rangle$;
(5) $\mathbf{e} = \langle 2, 2, 1\rangle$;
(6) $\mathbf{f} = \langle 4, 4, 2\rangle$;
(7) $\mathbf{g} = \langle 2, 6, 3\rangle$; and
(8) $\mathbf{h} = \langle -4, -12, -6\rangle$.

Problem 2: Solutions

(1) For $\mathbf{a} = \langle 3, 0, -4\rangle$, the unit vector is
$$\mathbf{a}/|\mathbf{a}| = \mathbf{a}/\sqrt{3^2 + 0^2 + (-4)^2} = \mathbf{a}/5 = \langle 3, 0, -4\rangle/5 = \langle 3/5, 0, -4/5\rangle;$$

(2) $\mathbf{b} = \langle 6, 0, 8\rangle$: we similarly compute
$$\mathbf{b}/|\mathbf{b}| = \langle 6, 0, 8\rangle/|\langle 6, 0, 8\rangle| = \langle 6, 0, 8\rangle/\sqrt{6^2 + 0^2 + 8^2} = \langle 6, 0, 8\rangle/10 = \langle 3/5, 0, 4/5\rangle;$$

(3) $\mathbf{c} = \langle -3, 0, 4\rangle$: we have $|\mathbf{c}| = \sqrt{(-3)^2 + 0^2 + 4^2} = 5$, and so
$$\mathbf{c}/|\mathbf{c}| = \langle -3, 0, 4\rangle/5 = \langle -3/5, 0, 4/5\rangle;$$

(4) $\mathbf{d} = \langle 3, 0, 4\rangle$: $|\mathbf{d}| = 5$ and so
$$\mathbf{d}/|\mathbf{d}| = \langle 3, 0, 4\rangle/5 = \langle 3/5, 0, 4/5\rangle;$$

(5) $\mathbf{e} = \langle 2,2,1\rangle$: $|\mathbf{e}| = \sqrt{2^2+2^2+1^2} = 3$, so
$$\mathbf{e}/|\mathbf{e}| = \langle 2/3, 2/3, 1/3\rangle;$$
(6) $\mathbf{f} = \langle 4,4,2\rangle$: $|\mathbf{f}| = 6$ and so
$$\mathbf{f}/|\mathbf{f}| = \langle 2/3, 2/3, 1/3\rangle;$$
(7) $\mathbf{g} = \langle 2,6,3\rangle$: $|\mathbf{g}| = \sqrt{4+36+9} = 7$ and so
$$\mathbf{g}/|\mathbf{g}| = \langle 2/7, 6/7, 3/7\rangle;$$
(8) $\mathbf{h} = \langle -4,-12,-6\rangle$: $|\mathbf{h}| = 14$ and so
$$\mathbf{h}/|\mathbf{h}| = \langle -2/7, -6/7, -3/7\rangle\ .$$

Problem 3

Based on your answer to Problem 2, answer the following questions.

(1) Which vectors in Problem 2 have the same direction?
(2) Which pairs vectors in Problem 2 have the opposite direction? [Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ have the opposite direction if $\mathbf{a}$ and $-\mathbf{b}$ have the same direction.]
(3) Which vectors in Problem 2 are parallel? [To be parallel is equivalent to having the same direction or the opposite direction.]

Problem 3: Solutions

(1) Comparing unit vectors: $\mathbf{b}$ and $\mathbf{d}$ have the same direction (since they have the same corresponding unit vectors); and $\mathbf{e}$ and $\mathbf{f}$ have the same direction.
(2) Those that have opposite unit vectors point in opposite directions, so $\mathbf{a}$ and $\mathbf{c}$ point in the opposite direction; and $\mathbf{g}$ and $\mathbf{h}$ point in the opposite direction.
(3) The following vectors have $\pm$ the same corresponding unit vectors and are therefore parallel: $\mathbf{a}$ and $\mathbf{c}$; $\mathbf{b}$ and $\mathbf{d}$; $\mathbf{e}$ and $\mathbf{f}$; and $\mathbf{g}$ and $\mathbf{h}$.

Problem 4

A computer is given 1000 3-dimensional vectors and must detect which are parallel. Consider the total number of operations—additions, subtractions, multiplications, divisions, and square roots, each counted as one operation—in the following two methods:

(1) (Using the ideas of Problems 2 and 3:) Find the associated unit vectors of the 1000 vectors. [You would still need to compare them, presumably by sorting the unit vectors, but let's ignore this cost.]
(2) Test all pairs of the 1000 vectors for being parallel, by the method of Problem 1. [There are 499,500 pairs of the 1000 vectors.]

Which method takes fewer operations? Would the difference be more significant with 1,000,000 vectors [there would be roughly 5.0×10^{11} pairs of vectors in this case]?

[As an example, the calculation

$$\sqrt{5^2 + (-4)^2} = \sqrt{5 \cdot 5 + (-4) \cdot (-4)}$$

requires one addition (+), two multiplications (two $\cdot$'s) and one square root, for a total of four operations.]

Problem 4: Solutions

(1) Finding the unit vector associated to $\langle 1, 2, 3\rangle$ requires computing

$$|\langle 1, 2, 3\rangle| = \sqrt{1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3}$$

using three multiplications, two additions, and one square root, for a total of six operations. Then finding the associated unit vector

$$\langle 1, 2, 3\rangle / |\langle 1, 2, 3\rangle| = \Big\langle 1/|\langle 1, 2, 3\rangle|, 2/|\langle 1, 2, 3\rangle|, 3/|\langle 1, 2, 3\rangle| \Big\rangle$$

takes an additional three divisions, for a total of 9 operations per unit vector computation. A similar 9 operations would be required for this computation for any of the 1000 vectors, for a total of 9000 operations.

(2) In Problem 1 we solve for c which requires three divisions (we ignore comparing the three values of c). Done for $499,500$ pairs of vectors gives close to 1.5 million operations.

So the second method requires (over 150) more times the cost, measured in operations. For 1,000,000 vectors the rough counts are 9 million versus some 1.5 trillion, a difference by a factor over 150,000.

Even if you counted the operations slightly differently, and got somewhat different operation counts, the bottom line is that the first method works on the individual vectors, while the second method works on pairs of vectors. So as the number of vectors gets large, the second method becomes much slower. If you had only 2 or 3 vectors, the story would be different...

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