Math 200, Nov. 16

§15.5 : Stop at centre of mass.

§15.6 : Omitted

§15.7 : Triple integrals

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2-dim: $\iint$, polar coordinates

3-dim: $\iiint$, $\begin{cases} \text{cylindrical coords} \\ \text{spherical coords} \end{cases}$

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Go back 2012WT1 final, problem 8

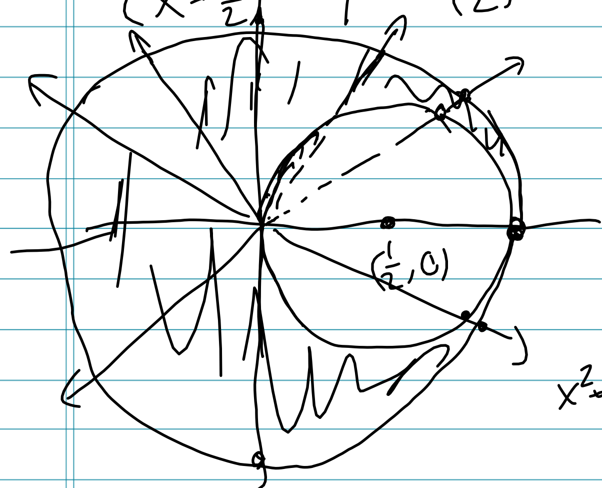
- $x^2+y^2 \le 1$
- removing interior of $x^2+y^2=x$

$x^2+y^2 = x \rightsquigarrow$

$$(x^2-x)+(y^2) = 0$$

$$x^2 - x + \frac{1}{4} + y^2 = \frac{1}{4}$$

$$\left(x-\frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2$$



$x^2+y^2 \le 1$

Area:

$90^\circ \le \theta \le 270^\circ \qquad 0 \le r \le 1$

$\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$

left part: $\displaystyle\int_{\theta=\pi/2}^{\theta=3\pi/2} \left(\int_{r=0}^{r=1} r\,dr \right) d\theta$

$$= \int_{\theta=\pi/2}^{\theta=3\pi/2} \underbrace{\left(\left(\frac{r^2}{2}\right) \Bigg|_{r=0}^{r=1} \right)}_{\frac{1^2}{2} - \frac{0^2}{2}} d\theta$$

$$\int_{\theta=\pi/2}^{\theta=3\pi/2} \frac{1}{2}\,d\theta = \frac{\theta}{2} \Bigg|_{\theta=\pi/2}^{\theta=3\pi/2}$$

$$= \frac{3\pi/2}{2} - \frac{\pi/2}{2} = \frac{\pi}{2}$$

=

$-90^\circ \le \theta \le 90^\circ$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$\int \left(\left(\int_{r=}^{r=1} r\,dr \right) d\theta \right)$$

we hit $(x^2+y^2=x)$

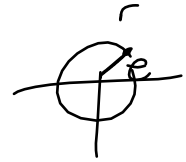
$$\int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \left(\frac{1^2}{2} - \frac{\cos^2\theta}{2}\right) d\theta$$

$$\int \left(\frac{1}{2} - \frac{\cos^2\theta}{2}\right) d\theta$$

$$= \frac{\theta}{2} - \int \frac{\cos^2\theta}{2}\, d\theta$$

[Hint: $\cos(2\theta) = \cos^2\theta - \sin^2\theta$
$= 1 - 2\sin^2\theta$
$= 2\cos^2\theta - 1$.]

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$



$$x = r\cos\theta$$
$$y = r\sin\theta$$

$=$

$$\underbrace{x^2+y^2}_{r^2} = x$$
$$r^2 = r\cos\theta$$

for us: $r = \cos\theta$

$$\int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \int_{r=\cos\theta}^{r=1} (r\,dr)\, d\theta$$

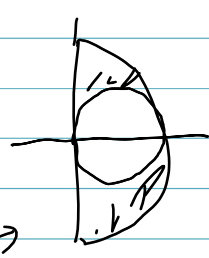
$$\int \left(\frac{r^2}{2} \Big|_{r=\cos\theta}^{r=1} \right) d\theta$$

$$\left(\frac{\pi/2}{4} - \frac{0}{4}\right) - \left(\frac{-\pi/2}{4} - \frac{0}{4}\right)$$

$$= \frac{\pi/2}{4} - \frac{-\pi/2}{4} = \frac{\pi}{4}$$

$=$

Method 2:
right half:



$\frac{1}{2}$ unit circle

$$\frac{1}{2}\cdot\pi - \pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{4}$$

$$\int \cos^2\theta\, d\theta = \int \left(\frac{1}{2} + \frac{\cos(2\theta)}{2}\right) d\theta$$

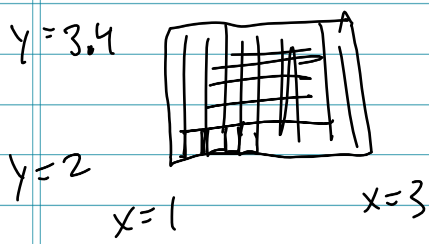
$$= \frac{\theta}{2} + \frac{\sin(2\theta)}{2\cdot 2}$$

$$\int_{\theta=-\pi/2}^{\theta=\pi/2} \left(\frac{1}{2} - \frac{\cos^2\theta}{2}\right) d\theta$$

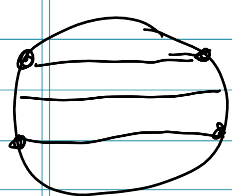
$$= \left[\frac{\theta}{2} - \frac{1}{2}\left(\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right)\right]_{\theta=-\pi/2}^{\theta=\pi/2}$$

$$= \left[\frac{\theta}{4} - \frac{\sin(2\theta)}{4}\right]_{\theta=-\pi/2}^{\theta=\pi/2}$$

Simple: Box



$$\int_{y=2}^{y=3.4} \left(\underbrace{\int_{x=1}^{x=3} f(x,y)\, dx}_{\text{fix } y} \right) dy$$



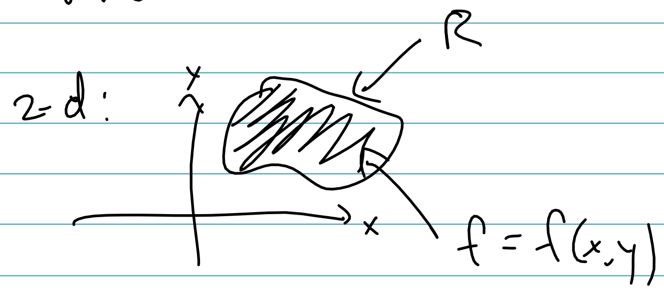
x = 3 ⇝ upper(y)

x = 1 ⇝ lower(y)

15.7 (no 15.6) (15.5 centre of mass)

$\iiint$ integrals

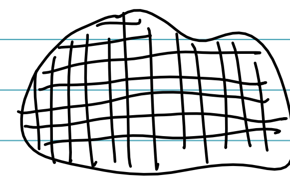
2-d:



$$\iint_R f(x,y)\, dA$$

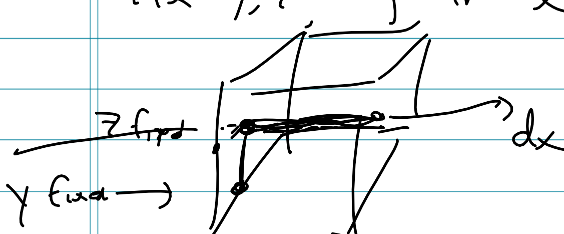
little area: $(\Delta x)(\Delta y)$

Theory:



add

Fix y, z $\int$ in x



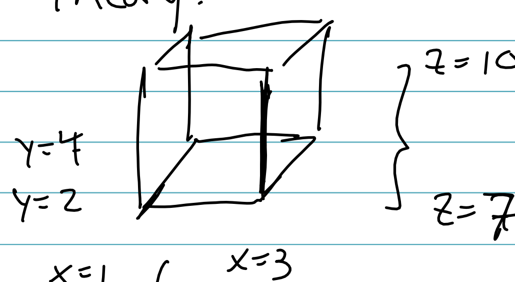
$$\iiint f = \int \left(\int \left(\underbrace{\int f(x,y,z)\, dx}_{\substack{y \text{ and} \\ z \text{ fixed}}} \right) dy \right) dz$$

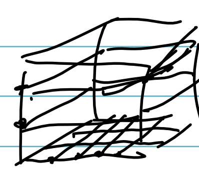
z fixed

Theory

$$\iiint_R f(x,y,z)\, dV =$$

Theory:





$$\iiint f(x,y,z)\, dV$$

small volume = $(\Delta x)(\Delta y)(\Delta z)$

$$\int_{z=3}^{z=7}\int_{y=2}^{y=4}\int_{x=0}^{x=1} f(x,y,z)\,dx\,dy\,dz = \int_{z=}^{z=}\left(\int_{y=}^{y=}\left(\int_{x=}^{x=} f(x,y,z)\,dx\right)dy\right)dz$$

$$\iiint \quad dy\,dx\,dz$$

$$= \iiint \quad dx\,dz\,dy$$

$$= \iiint \quad dz\,dx\,dy$$

$$= \iiint \quad dz\,dy\,dx$$

$$= \iiint \quad dy\,dz\,dx$$

$=$

Say region in 3-dim:

$$\left.\begin{array}{l} 0 \le x \le 1 \\ 2 \le y \le 4 \\ 3 \le z \le 7 \end{array}\right\} \quad \iiint f(x,y,z)\,dV$$