Math 200, Oct 14

14.3, 14.4 preliminary to 14.5, 14.6, ... ← old final exam problems

14.5: Chain rule (implicit diff)

14.6: Gradient (NEW)

2012 WT1, Problem 3

$z = f(x,y)$, find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ ...

implicitly:

$$xyz + x + y^2 + z^3 = 0$$

z ... cubic equation ...

$\frac{\partial z}{\partial x}$ = what is $\boxed{\frac{\partial}{\partial x} = \begin{cases} \text{derivative in } x \\ \text{hold } y \text{ fixed} \end{cases}}$

$$\frac{\partial}{\partial x}\left(xyz + x + y^2 + z^3 = 0 \right)$$

holding y fixed

$$\frac{\partial}{\partial x}(xyz) = y \frac{\partial}{\partial x}(xz)$$

$$= y\left(\frac{\partial x}{\partial x} \cdot z + x \frac{\partial z}{\partial x} \right)$$

$$= y\left(1 \cdot z + x \frac{\partial z}{\partial x} \right)$$

$$\frac{\partial}{\partial x}\left(xyz + x + y^2 + z^3 \right)$$

$$= y\left(z + x\frac{\partial z}{\partial x}\right) + 1 + 0 + 3z^2 \frac{\partial z}{\partial x}$$

Find $\frac{\partial z}{\partial x}$:

$$\frac{\partial}{\partial x}\left(xyz + x + y^2 + z^3 = 0 \right)$$

$$yz + xy\frac{\partial z}{\partial x} + 1 + 0 + 3z^2 \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x}\left(xy + 3z^2\right) = -(1 + yz)$$

$$\frac{\partial z}{\partial x} = \frac{-(1+yz)}{xy + 3z^2}$$ involves z

Recall: $x^2 + y^2 = 1$, $\frac{dy}{dx} = ?$

$$y = \pm\sqrt{1-x^2} = (1-x^2)^{1/2},$$

$$\boxed{y'} = \frac{dy}{dx} = \frac{d}{dx}(1-x^2)^{1/2}$$

$$\boxed{= (1-x^2)^{-1/2} \tfrac{1}{2}(-2x)}$$

Easier (?):

$$\frac{d}{dx}\left(x^2 + y^2 = 1 \right)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = \frac{-2x}{2y} = -x/y}$$

implicit

implicit

$$\frac{\partial z}{\partial y}\left(xy+3z^2\right) = -(xz+2y)$$

$$\frac{\partial z}{\partial y} = -\frac{xz+2y}{xy+3z^2}$$

$=$

What is $f(-1,1) = z$ at $x=-1$, $y=1$

???

$xyz + x + y^2 + z^3 = 0$

at

$x=-1,\ y=1$

$$(-1)(1)z + (-1) + (1)^2 + z^3 = 0$$

$$\boxed{z^3 = z \ :\ z = 0, 1, -1}$$

Part (a):

$$\frac{\partial z}{\partial x} = \frac{-(1+yz)}{xy+3z^2}.$$

Δf, $\boxed{f = f(x,y) = z}$

$$\Delta f = \Delta z = f_x \Delta x + f_y \Delta y$$

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

$=$

$$\frac{\partial z}{\partial y}:\ \frac{\partial}{\partial y}\left(xyz + x + y^2 + z^3 = 0\right)$$

$$\Rightarrow \underbrace{x\frac{\partial}{\partial y}(yz)} + 0 + 2y + 3z^2\frac{\partial z}{\partial y} = 0$$

$$x\left(1\cdot z + y\frac{\partial z}{\partial y}\right) + 2y + 3z^2\frac{\partial z}{\partial y} = 0$$

$$\Delta z \approx z_x \Delta x + z_y \Delta y$$

Part (ii) $\boxed{= 0\,\Delta x + \left(\frac{-3}{2}\right)\Delta y}$

near $x=-1$: $x_{near} = -1.02$

$y=1$: $y_{near} = 0.97$

$\Delta x = -(.02)$

$\Delta y = -(.03)$

Δz approx: $z_x \Delta x + z_y \Delta y$

Part (iii)

$$= 0\cdot(-0.02) + \left(\frac{-3}{2}\right)(-0.03)$$

$$= 0 + \frac{0.09}{2} = 0.045$$

Look at: $(x,y,z) = (-1,1,-1)$

3 (ii), (iii)

What is linear approx of $z = f(x,y)$

near there, approx $f(-1.02, 0.97)$

$$\frac{\partial z}{\partial x} = \frac{-(1+yz)}{xy+3z^2}, \quad \frac{\partial z}{\partial y} = \frac{-(xz+2y)}{xy+3z^2}$$

Near

$x=-1,\ y=1,\ z=-1$

$\frac{\partial z}{\partial x}$ at $(-1,1,-1)$ is $\dfrac{-(1+(-1))}{-1+3(-1)^2} = 0$

$\frac{\partial z}{\partial y}$ at $(-1,1,-1)$ is $\dfrac{-(1+2)}{(-1)+3(-1)^2} = \dfrac{-3}{2}$