CPSC 536J NOTES AND EXERCISES

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ABSTRACT. Here is an abstract of the paper.

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THIS ARTICLE IS CURRENTLY IN DRAFT FORM. SOME MISTAKES IN THIS ARTICLE WILL ONLY BE CORRECTED IN CLASS.

1. INTRODUCTION

CPSC 536J for Spring 2019 focuses on applications of linear algebra to computer science.

2. OUTLINE

In this section we outline some class discussion.

2.1. **Review of Eigenvalues.** If $A \in \mathbb{R}^{n\times n}$, we say that $(\lambda, \mathbf{v})$ is an *eigenpair* for A if $\mathbf{v} \neq 0$ and $A\mathbf{v} = \lambda v$; λ is called an *eigenvalue* of A, and $\mathbf{v}$ an *eigenvector*. In this case

$$E_\lambda \stackrel{\text{def}}{=} \ker\bigl((A - I\lambda)^n\bigr)$$

(where I denotes the identity matrix) is called the *generalized eigenspace* associated to λ. Each vector in $\mathbb{R}^n$ can be written uniquely as a linear combination of vectors in the $E_\lambda = E_\lambda(A)$, and $\dim(E_\lambda)$ is the multiplicity of λ as a root of the *characteristic polynomial of* A, i.e., the polynomial $p_A(x) = \det(xI - A)$.

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Example 2.1. Consider

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

Then we easily compute

$$p_A(x) = \det(xI - A) = x^2, \quad p_B(x) = (x-3)^2.$$

Since $A^2 = 0$ (where 0 is the all zeros matrix), we have $E_0(A) = \mathbb{R}^2$; similarly $E_3(B) = 0$.

Example 2.2. Say that D is a *diagonal matrix* with diagonal entries $d_1, \ldots, d_n$ (meaning that $D_{i,j} = 0$ if $i \neq j$ and $D_{i,i} = d_i$). Then D has eigenpairs $(d_i, \mathbf{e}_i)$, where $\mathbf{e}_i$ is the *i-th element of the standard basis for* $\mathbb{R}^n$, i.e., the vector that is zero at all components except the i-th component, whose value is 1. If p is any polynomial, then we have that $p(D)$ is the diagonal matrix with diagonal entries $p(d_1), \ldots, p(d_n)$; if f is any function, it often makes sense to define $f(D)$ to be the diagonal matrix with entries $f(d_1), \ldots, f(d_n)$ (e.g., f has a globally convergent power series, such as $f(x) = \sin(x)$ or $f(x) = e^x$).

Example 2.3. If A, B are $n \times n$ matrix, then we say that A and B are *similar* if $B = M^{-1}AM$ for some invertible $n \times n$ matrix M. In this case A, B have the same eigenvalues and $p(B) = M^{-1}p(A)M$ for any polynomial, p.

If $A = A^T$ is symmetric, then

(1) all eigenvalues of A are real;
(2) if $(\lambda, \mathbf{v})$ and $(\mu, \mathbf{u})$ are eigenpairs of A with $\lambda \neq \mu$, then $\mathbf{v} \cdot \mathbf{u} = 0$;
(3) there is an *orthonormal eigenbasis* for A, meaning an orthonormal set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ spanning $\mathbb{R}^n$ and real $\lambda_1, \ldots, \lambda_n$ such that $(\lambda_i, \mathbf{v}_i)$ is an eigenpair for A for all i; in this case

$$A = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i \mathbf{v}_i^{\mathrm{T}}.$$

Example 2.4. Let $\vec{K}_n$ denote the complete directed graph on n vertices $V = [n] \stackrel{\text{def}}{=} \{1, 2, \ldots, n\}$ with one directed edge from each vertex to each other (including each vertex to itself). The adjacency matrix of $\vec{K}_n$ is the all 1's matrix A, i.e.,

$$A = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{bmatrix}$$

whose eigenvalues are n with multiplicity 1 and 0 with multiplicity $n-1$; the eigenvalue $\lambda = n$ has eigenvector $\mathbf{1}$ (the all 1's vector), which when normalized can be taken to be $\mathbf{v}_1 = \mathbf{1}/\sqrt{n}$. We have

$$\mathbf{v}_1(\mathbf{v}_1)^{\mathrm{T}} = \begin{bmatrix} 1/n & 1/n & \ldots & 1/n \\ 1/n & 1/n & \ldots & 1/n \\ \vdots & \vdots & \ddots & \vdots \\ 1/n & 1/n & \ldots & 1/n \end{bmatrix} = \frac{1}{n}A$$

is the projection onto the vector $\mathbf{1}$ (or, equivalently $\mathbf{v}_1$), and

3. Graph Adjacency Eigenvalues: Examples and Exercises

In all these exercises, eigenvalues, eigenvectors, and eigenpairs refer to those of the associated adjacency matrix.

Definition 3.1. If $\mathcal{G}$ is a group and $\{g_1, \ldots, g_d\}$ a subset (or multiset) of elements of $\mathcal{G}$, we use

$$G = \mathrm{Cayley}(\mathcal{G}, \{g_1, \ldots, g_d\})$$

to denote the graph whose vertex set is $V_G = \mathcal{G}$, with a directed edge (g, gg_i) for all $g \in \mathcal{G}$ and $i \in [d]$ (which can be multiedges).

Example 3.2. If $\mathcal{G} = \mathbb{Z}/n\mathbb{Z}$ is the group of integers modulo n (i.e., the cyclic group of order n), then for each $\zeta \in \mathbb{C}$ with $\zeta^n = 1$, there is an eigenfunction $f_\zeta \colon \mathcal{G} \to \mathbb{C}$ given by $f_\zeta(i) = \zeta^i$ is an eigenfunction with eigenvalue

$$\lambda_\zeta = \sum_{i \in [d]} \zeta^{g_i}. \tag{1}$$

Endowing $\mathbb{C}^n$ with the dot product

$$\langle f, h \rangle \stackrel{\text{def}}{=} \sum_{i \in [n]} \overline{f(i)} h(i).$$

Exercise 3.1. Let $p \equiv 1 \pmod 4$. Let Q be the set of quadratic residues modulo p.

3.1(a) Show that for any ζ with $\zeta^p = 1$ and $\zeta \neq 1$ we have

$$\lambda_\zeta = \frac{\pm\sqrt{p} - 1}{2}.$$

[Hint: we have that

$$2\lambda + 1 = \sum_{a=0}^{p-1} \zeta^{a^2},$$

and

$$(2\lambda + 1)^2 = \sum_{a,b=0}^{p-1} \zeta^{(a+b)(a-b)}$$

which after a change of variables is

$$\sum_{u,v=0}^{p-1} \zeta^{uv}.$$

Consider the sum over all v above with $u = 0$ and for each with $u \neq 0$.]

3.1(b) For how many of the $p - 1$ values of ζ with $\zeta^p = 1$ and $\zeta \neq 1$ do we have

$$\lambda_\zeta = \frac{\sqrt{p} - 1}{2}?$$

See the bottom for a hint[1]

3.1(c) What are the eigenvalues of $\mathrm{Cayley}(\mathbb{Z}/p\mathbb{Z}, Q)$ (and what are their multiplicities)?

3.1(d) Same question with Q replaced with the quadratic nonresidues.

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[1] Consider the trace of an appropriate matrix.

Exercise 3.2. Let G, H be graphs with adjacency matrices A_G, A_H. Let $G \times H$ be the graph with vertex sets $V_G \times V_H$ and with an edge (v_1, u_1) to (v_2, u_2) if EITHER $v_1 = v_2$ and H has an edge (u_1, u_2) OR $u_1 = u_2$ and G has an edge (v_1, v_2).

3.2(a) If G is d_1-regular and H is d_2-regular, what is the regularity of $G \times H$?
3.2(b) Find the eigenvalues and eigenvectors of the adjacency matrix $A_{G\times H}$ in terms of those of A_G, A_H.

Exercise 3.3. Let G, H be graphs with adjacency matrices A_G, A_H. Let $G \otimes H$ be the graph with vertex sets $V_G \times V_H$ and with an edge (v_1, u_1) to (v_2, u_2) if H has an edge (u_1, u_2) AND and G has an edge (v_1, v_2).

3.3(a) If G is d_1-regular and H is d_2-regular, what is the regularity of $G \otimes H$?
3.3(b) Find the eigenvalues and eigenvectors of the adjacency matrix $A_{G\otimes H}$ in terms of those of A_G, A_H.

Exercise 3.4. Let $\mathbb{B}$ denotes the graph with vertex set $V_{\mathbb{B}} = \{0, 1\}$ with a single edge from 0 to 1; let $\mathbb{B}^n = \mathbb{B}^{\times n}$ (i.e., the n-fold product $\mathbb{B} \times \cdots \times \mathbb{B}$). Determine the eigenpairs of $\mathbb{B}^n$.

Exercise 3.5. Let P_n denote the graph that is commonly called the *path of length* n, i.e., the graph with vertex set $[n] = \{1, \ldots, n\}$, and with an edge joining $i, j \in V$ iff $|i - j| = 1$. Determine a set of eigenvectors and eigenvalues for P_n as follows:

3.5(a) By a directed eigenvalue computation, find the eigenpairs for P_n for $n = 1, 2, 3$.
3.5(b) Determine the eigenpairs for the cycle of length $2n + 2$, viewed as $\mathrm{Cayley}(\mathbb{Z}/(2n + 2)\mathbb{Z}, \{\pm 1\})$.
3.5(c) For $f : \mathbb{Z}/(2n+2)\mathbb{Z} \to \mathbb{C}$, let σf be the function $\mathbb{Z}/(2n+2)\mathbb{Z} \to \mathbb{C}$ given by $(\sigma f)(i) = f(-i)$; say that such a function is *odd* if $\sigma f = -f$. Show that an eigenfunction of the cycle (i.e., the adjacency matrix of the cycle) that is odd restricts to an eigenfunction of the path, whose vertex set $[n]$ is viewed as a subset in $\mathbb{Z}/(2n + 2)\mathbb{Z}$ in the evident (quotient) fashion.
3.5(d) Find a formula for the eigenpairs of P_n using the previous part.
3.5(e) Check the formula of the previous part for $n = 1, 2, 3$ and the computations in part (a).
3.5(f) Let G be a graph, each of whose vertices are of degree d or $d/2$ for some even integer $d \geq 2$. Let $\mathrm{Double}(G)$ be the graph consisting of two disjoint copies of G where each vertex of degree $d/2$ is connected to its "mirror vertex" (i.e., the same vertex in the other copy) by a single edge. Define a notion of *odd eigenfunction* and generalize the remark in the previous parts of this exercise to relate odd adjacecy eigenfunctions of $\mathrm{Double}(G)$ and those of G.
3.5(g) If we allow multiple edges in our graphs, can we generalize the last part further? [For example, say that you can add self-loops of any degree (odd degrees are OK) to any vertex (we will do this later in the course). If G is a graph each of whose vertices are of degree at most d, is there a $\mathrm{Double}(G)$ that is a $2(d-1)$ regular graph such that the odd eigenpairs of $\mathrm{Double}(G)$ can be related to those of G?

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