Class Notes Starting Jan 2

CPSC 536 J. Applied Linear Algebra

- Starting Monday, Jan 7, class 8:45 – 9:35 MWF
- Questionnaire on Friday regarding topics & times & grading
- Meaning of grades:

95% : Very Strongly to work in the area, promises that the instructor writes a strong letter of recommendation

90% : Strong ---

85% : $\begin{cases} \text{You need more basics to work in this field} \\ \text{You learned a reasonable amount to use in another field} \end{cases}$

80% : You have fulfilled the expectations of a grad student

< 80% : " " not " " " " " " "

Grades: Homework problems, collected $\begin{cases} \text{middle of term} \\ \text{end of term} \end{cases}$

Work on a project related to the course
(1) pure research + expository article
(2) applied research, computations
(3) present to class some topic

First few weeks: what do the first few eigenvalues tell us and don't tell us about large systems?

- First topic:
  - Symmetric matrices: $A^T = A$, more generally
  - self-adjoint operators $A^* = A$ on inner product spaces
- Abstract Theorem: Such matrices/operators have an orthonormal eigenbasis
- Examples:
  - Expanding graphs $[A = \mathrm{Adj}\ \mathrm{mat} : A^T = A]$
  - Reversible Markov chains
  - FA (factor analysis) $\approx$ PCA (principle component analysis)

    $\approx$ SVD (singular-value decomposition)
  - Variance-Covariance
  - Laplacians of: graphs, simplicial complexes, etc.

positive semi-definite, $M^T M$

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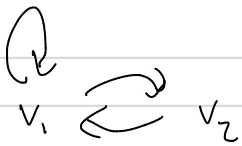
Example:

Fibonacci graph:



(directed graph, we allow self-loops, multiple edges)

Adjacency matrix (Fib graph):
$$\begin{array}{cc} & \begin{array}{cc} v_1 & v_2 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \end{array} & \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \end{array} = A$$

$(Adj)_{u,v} = \left\{ \begin{array}{l} \# \text{ edges from} \\ u \text{ to } v \end{array} \right.$

$$A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}, \quad A^4 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}, \ldots$$

$F_1 = 1, \ F_2 = 1, \ F_3 = 2, \ldots \qquad F_n = F_{n-1} + F_{n-2}$

$$A \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} F_n + F_{n-1} \\ F_n \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$$

Diagonalize: 2 eigenvalues $\frac{1+\sqrt{5}}{2}$, $\frac{1-\sqrt{5}}{2}$

$\frac{1+\sqrt{5}}{2} \to 1.6\ldots$

$\frac{1-\sqrt{5}}{2}$: between $-1, 0$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} = \left(\frac{1+\sqrt{5}}{2}\right) \begin{bmatrix} \\ \\ \end{bmatrix}}_{\text{same}}$$

We'll see

$F_n =$ nearest integer to $\left(\frac{1+\sqrt{5}}{2}\right)^n \frac{1}{\sqrt{5}}$

Quest: $F_{-1}, F_{-2}, \ldots$

Jan 4

- Today $\to$ 9:30 am
- Start next week 8:50 am $\hookleftarrow$ MWF
  8:45 am

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Operators: $T : V \to V$ $\begin{cases} \leftarrow \text{inner product, dot product} \\ \leftarrow V = \mathbb{R}^n \end{cases}$

$\mathbb{R}^n$ for now

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$T : \mathbb{R}^n \to \mathbb{R}^n$ think of it: standard basis $e_1 = (1, 0, 0, \ldots)$
$\vdots \quad (0, 1, 0, \ldots)$
$e_n \quad \vdots$

$T \rightsquigarrow$ matrix $\underbrace{\begin{bmatrix} & & \end{bmatrix}}_{n} \Big\} n$

eigen values vectors

Abstractly: $(\lambda, \vec{v})$ eigenpair for $A \in \mathbb{R}^{n \times n}$ if

$A\vec{v} = \lambda \vec{v}$, $\vec{v} \neq 0$.

If λ is eigenvalue, $E_\lambda = \ker(A - \lambda I)^n$

generalized eigenspace of λ

Thm: Each vector $\vec{u} \in \mathbb{R}^n$ can be written uniquely as

Fix $A \in \mathbb{R}^{n \times n}$.

$$\vec{u} = \sum_{\lambda \in \mathrm{Eigs}(A)} \vec{u}_\lambda, \quad \vec{u}_\lambda \in E_\lambda = E_\lambda(A)$$

Examples:

(0) Last time $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ adj mat of Fibonacci graph,

$$\lambda = \frac{1 \pm \sqrt{5}}{2}$$

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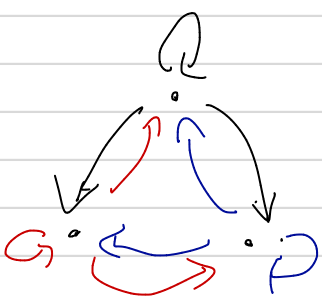
(1) Complete digraph on 3 vertices



$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

λ's : $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

λ's : 0's

because: C. : $A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ rank 1

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} + 0$$

$E_0 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| \; a+b+c = 0 \right\}$ = Vects $\perp$ to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, i.e. $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$

$A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b+c \\ a+b+c \\ a+b+c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad \{ v \mid A\vec{v} = 0 \cdot \vec{v} \}$

Last time: Any symmetric A $(A^T = A)$ has orthonormal eigenbasis $\vec{v}_1, \dots, \vec{v}_n$ s.t. $A\vec{v}_i = \lambda_i \vec{v}_i$

and $\vec{v}_i \cdot \vec{v}_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

then

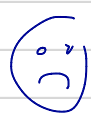
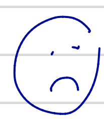
$$A = \sum_{i=1}^{n} \lambda_i \underbrace{\left(\vec{v}_i \, \vec{v}_i^{\,T}\right)}_{\text{rank } 1}$$

=

e.g.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \lambda_1 = 3, \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} / \sqrt{3} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$\lambda_2, \lambda_3 = 0, \qquad \vec{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}$$

$$A = 3 \cdot \vec{v}_1 \vec{v}_1^{\,T} + 0 \cdot \vec{v}_2 \vec{v}_2^{\,T} + 0 \cdot \vec{v}_3 \vec{v}_3^{\,T}$$

=

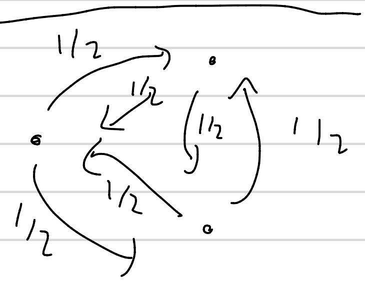
$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = A - I = \mathrm{adj}_{\triangle} = 2 \, \vec{v}_1 \vec{v}_1^{\,T} + (-1) \, \vec{v}_2 \vec{v}_2^{\,T} + (-1) \, \vec{v}_3 \vec{v}_3^{\,T}$$

Question: Is there a nice way to write

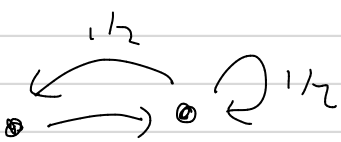
$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = (2)\boxed{?} + (-1)\underbrace{\boxed{?}}_{\substack{\text{Proj onto orthog. comp} \\ \text{of } \begin{bmatrix}1\\1\\1\end{bmatrix}}}$$

$$= E_{-1}\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\right)$$

$$= E_0\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right)$$



$$\begin{bmatrix} 0 & \frac12 & \frac12 \\ \frac12 & 0 & \frac12 \\ \cancel{0}\,\frac12 & \frac12 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 0 & 1 \\ \frac12 & \frac12 \end{bmatrix}$$

$\boxed{\text{Why do we write } A^* = A \text{ vs. } A^T = A}$

Jan 7:

If $A \in \mathbb{R}^{n\times n}$, i.e. A is $n\times n$ real matrix:

<u>eigenpair</u> for A: $(\lambda, \vec{v})$ s.t. $A\vec{v} = \lambda\vec{v}$

eigenvalue eigenvector

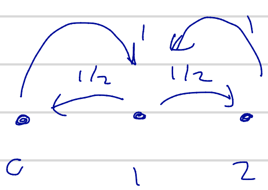
First few weeks: A think of $n\times n$, n large

$A \leftrightarrow$ local structure: if eigenvalues of A: $\lambda_1, \dots, \lambda_n$

have, say $\lambda_1, \lambda_2, \lambda_3$ big, $\lambda_4, \dots \lambda_n$ small(er), can we say useful things about A?

Examples: (1) Adjacency matrices of $\underbrace{\text{graphs}}_{\text{symmetric matrices}}$ / digraphs

(2) Markov chains:



$p_{0\to 1} = 1 = p_{2\to 1}$

$p_{1\to 2}, p_{1\to 0} = 1/2$

Markov matrix is one with stochastic rows: $\underline{P} = \begin{bmatrix} p_{00} & p_{01} & p_{02} \\ p_{10} & p_{11} & p_{12} \\ p_{20} & p_{21} & p_{22} \end{bmatrix}$

$p_{21} = p_{2\to 1}$ $\underline{P} = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$

Markov chains: (1) What happens to $\underline{P}^1, \underline{P}^2, \dots, \underline{P}^n$, $n\to\infty$

(2) For what n is $\underline{P}^n$ close to $\underline{P}^\infty := \lim_{n\to\infty} \underline{P}^\infty$?

(3) Often matrices $A \in \mathbb{R}^{n\times n}$ with $A^T = A$ (symmetric)
(SVD, F.A., PCA, Laplacians, ...)

General Theory: $A \in \mathbb{R}^{n \times n}$

identity matrix

$$A\vec{v} = \lambda \vec{v}, \quad \vec{v} \neq 0 \iff A\vec{v} = \lambda I \vec{v}$$

$$\iff (A - \lambda I)\vec{v} = 0$$

λ eigenvalue of A $\iff$ $A - \lambda I$ $\begin{cases} \text{- is not invertible} \\ \text{- has non-zero kernel/nullspace} \\ \text{- has zero determinant} \\ \vdots \end{cases}$

$\iff$ $\det(xI - A) = \underset{\text{poly}}{\text{char}}(A)(x)$ has root λ.

Thm: $\det(xI - A) = p_A(x)$ has n roots, and if λ is a root of multiplicity k, then $E_\lambda(A) := \ker(\lambda I - A)^n$ has dim k.

E.g. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\det(xI - A) = \det\begin{bmatrix} x & -1 \\ 0 & x \end{bmatrix} = x^2$

λ's of A: $0, 0$. $\lambda I - A \Big|_{\lambda = 0} = -A = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$ ☹

$\{v \mid A\vec{v} = 0 \cdot \vec{v}\} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

$\{\vec{v} \mid (A - I \cdot 0)^{\text{pos power}} \vec{v} = \vec{0}\}$: $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

"defect of $\lambda = 0$" $\iff$ $\ker(I\lambda - A) = \text{Eigenvectors}_{\lambda=0}$ is 1-dimensional e.g. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = A + 3I = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$

$$A\vec{v} = \lambda\vec{v} \iff B\vec{v} = (\lambda+3)\vec{v}$$

matrix $\begin{bmatrix} \lambda & 1 & & 0\text{'s} \\ & \lambda & 1 & \\ & & \ddots & 1 \\ 0\text{'s} & & & \lambda \end{bmatrix}$ "Jordan block"

Any matrix is similar to a block diagonal matrix whose blocks are Jordan blocks

$$\begin{array}{c} 2\{ \\ 2\{ \\ 1\{ \end{array} \left[\begin{array}{c|c|c} \begin{matrix} 3 & 1 \\ 0 & 3 \end{matrix} & 0\text{'s} & 0\text{'s} \\ \hline 0 & \begin{matrix} 5 & 1 \\ 0 & 5 \end{matrix} & 0 \\ \hline 0 & 0 & 2 \end{array}\right]$$
$$\quad \underbrace{\quad}_{2} \quad \underbrace{\quad}_{2} \quad \underbrace{\quad}_{1}$$

A, B $n \times n$ are similar if $B = M^{-1}AM$

$(A \sim B)$ for M invertible

① $A \sim B \iff A, B$ differ by a change of basis

$B = M^{-1}AM$ (we understand A) $\implies B^{100} = \underbrace{M^{-1}AM}\,\underbrace{M^{-1}AM} \cdots M^{-1}AM$

$= M^{-1}A^{100}M$

$$B = M^{-1} A M \qquad (\mathbb{R}^{n\times n})$$

p = polynomial $\qquad p(x) = 2x^2 + x^4$

$$p(B) = 2B^2 + B^4$$

$$= 2(M^{-1}AM)^2 + (M^{-1}AM)^4 = M^{-1}(2A^2 + A^4)M$$

$$p(B) = M^{-1}\, p(A)\, M$$ ← for any reasonable p

If $A = \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \\ & & & d_n \end{bmatrix}$, $\quad A^2 = \begin{bmatrix} d_1^2 & & \\ & d_2^2 & \\ & & \ddots & \\ & & & d_n^2 \end{bmatrix}$

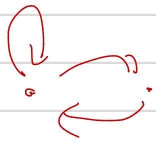
$$p(A) = \begin{bmatrix} p(d_1) & & \text{0's} \\ & \ddots & \\ \text{0's} & & p(d_n) \end{bmatrix}$$

<u>Def</u>: B is <u>diagonalizable</u> iff $B = M^{-1}\begin{bmatrix} d_1 & & \text{0's} \\ & \ddots & \\ \text{0's} & & d_n \end{bmatrix} M$

Then $p(B) = M^{-1}\begin{bmatrix} p(d_1) & & \text{0's} \\ & \ddots & \\ \text{0's} & & p(d_n) \end{bmatrix} M$ ←

Will see: if $B^T = B$, M is <u>orthogonal matrix</u>, things work nicely; otherwise M, M^{-1} "not nice"

Examples:

Fibonacci graph  adjacency mat $\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{}^{A}$ Symm

$$A = m^{-1} \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} m \qquad d_1, d_2 = \frac{1 \pm \sqrt{5}}{2}$$

A^n, n large $\approx m^{-1} \begin{bmatrix} d_1^n & 0 \\ 0 & \text{close to } 0 \end{bmatrix} m$

A^n, n ~~very~~ negative $\approx m^{-1} \begin{bmatrix} \text{close to } 0 & 0 \\ 0 & d_2^n \end{bmatrix}$

eigenvectors: $\begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$ $\Leftarrow r^2 = r + 1$

$\Leftrightarrow F_n = F_{n-1} + F_{n-2}$

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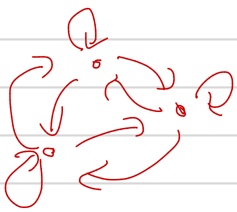
Recall: If $A^T = A$, then $A\vec{v} = \lambda \vec{v}$, $A\vec{u} = \mu \vec{u}$, $\lambda \neq \mu$, $\vec{v} \cdot \vec{u} = 0$

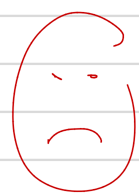
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad \lambda = 3, 0, 0$$

$$\longleftarrow \quad \sum_{\lambda \in \text{eigs}(A)} \lambda \cdot (\text{proj onto eigenspace of } \lambda)$$



$$\lambda_1 = 3, \quad \vec{v}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$\lambda_2 = 0, \quad \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$$



$$\lambda_3 = 0, \quad \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}$$

if orthonormal $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$

$$A\vec{v}_i = \lambda_i \vec{v}_i \qquad \sum_i \lambda_i (\vec{v}_i \vec{v}_i^T)$$

Next time:

- Markov matrices

- Matrices like $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $\qquad \lambda = 3, 0, 0$

expansion bounds

connected?

and matrices of d-regular graphs

connectivity?

Jan 9: Today: Edge expansion, Markov chains

Questions (courtesy of Carl (et al.?)):

(1) $\sin(A) = A - \frac{A^3}{3!} + \frac{A^5}{5!} - \cdots$ what norm? (Short answer: doesn't matter)

(2) If $A \neq 0$ is nilpotent, is it not diagonalizable? (Short answer: yes)

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If $A \in \mathbb{R}^{m \times n}$, typical norms:

(1) $\|A\|_{Frob} = \sqrt{\mathrm{Tr}(A^T A)} = \sqrt{\sum_{i,j} (A_{ij})^2}$ entries of A

[$\mathrm{Tr}(M)$, M square, $\mathrm{Tr}(M) = \sum_i M_{ii} = \sum_{i=1}^n \lambda_i$, ...
$\mathrm{Tr}(A^T B) = \mathrm{Tr}(B^T A)$, ...]

Operator norm: $\mathcal{L}: V \to V$, some norm on V,

Norm: $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ ($\mathbb{R}$-vector sp, $\mathbb{C}$- " ")

(1) $\|\alpha \vec{v}\| = |\alpha| \, \|\vec{v}\|$

(2) $\|\cdot\|$ is a metric: (1) $\|v\| \geq 0$ and eq iff $v = 0$

(2) $\|v + w\| \leq \|v\| + \|w\|$

$\rho_{\|\cdot\|}(v,u) := \|v - u\|$

(2) $\|\mathcal{L}\|_{\text{wrt some } \|\cdot\| \text{ on } V} := \max_{v \neq 0} \frac{\|\mathcal{L} v\|}{\|v\|}$

Remark: If V is a finite dim $\mathbb{R}$-vector space ($\mathbb{C}$-vector space)

Then for any norms $\| \ \|_1$, $\| \ \|_2$, there are C', C st.

for all $v \in V$

$$0 < C' \le \frac{\|v\|_{\text{first}}}{\|v\|_{\text{second}}} \le C$$

(C', C depend on $\| \ \|_1$, $\| \ \|_2$)

So $v_1, v_2, v_3, \ldots \in V$:

$$\lim_{i \to \infty} \|v_i\|_{\text{first}} = 0 \iff \lim_{i \to \infty} \|v_i\|_{\text{second}} = 0$$

Proof: Take a basis for $V \leftrightarrow \mathbb{R}^n$

$v_1, \ldots, v_n$

$\Rightarrow$ each elt of V is uniquely rep as

$\alpha_1 v_1 + \ldots + \alpha_n v_n \rightarrow \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$

$$\boxed{\sqrt[3]{|\alpha_1|^3 + |\alpha_2|^3 + \ldots + |\alpha_n|^3} = 27.715}$$

supremum / infimum

max/min of $\frac{\|v\|_{\text{first}}}{\|v\|_{\text{second}}}$ is same as max/min on $\left\{ v \sim \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \;\middle|\; \underbrace{\alpha_1^2 + \ldots + \alpha_n^2 = 1}_{\text{compact, non-zero vectors}} \right\}$

continuous in V

Infimum (greatest lower bound) is attained at some vector $u \in V$

Hence $u \neq 0$, $0 \neq \frac{\|u\|_1}{\|u\|_2} \le$ all $\frac{\|v\|_1}{\|v\|_2}$

Remark: For all $v \in \tilde{V}$

$$\frac{\|v\|_{first}}{\|v\|_{third}} = \frac{\|v\|_{first}}{\|v\|_{second}} \; \frac{\|v\|_{second}}{\|v\|_{third}}$$

So

$$\frac{\|v\|_{first}}{\|v\|_{third}} \leq \sup\left(\qquad \right) \sup\left(\qquad \right)$$

So

$$\sup\left(\qquad \right) \leq \quad '' \quad ''$$

$=$

If $A \neq 0$ but A nilpotent $\Rightarrow$ λ's of $A = 0$

(exercise)

but

$$A \neq \underbrace{M^{-1} \, \text{Zeros} \, M}_{\text{Zeros}}$$

$=$

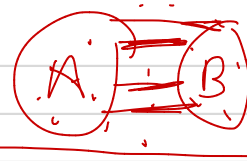
d-reg graph

 $\Big\}$ d neighbors,

$$A = \begin{bmatrix} 0\,0\,1\,1\,0\,1\,0\,0 \\ \cdots \\ \ddots \end{bmatrix}$$

col sums
row sums
$= d$

Edge expansion:

Graph: 

$E(A,B) :=$ # edges from A to B

$n = \#V = |V|$

Thm: Let G be a d-regular graph on n vertices. (so $d = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq -d$). If $A, B \subset V$, then

$$\left| \left(\#\text{ edges from } A \text{ to } B \right) - \frac{d}{n} |A|\,|B| \right|$$

not tight

$$\leq \rho \sqrt{\frac{|A|(n-|A|)}{n}} \sqrt{\frac{|B|(n-|B|)}{n}}$$

OK for $|A|, |B|$ small but $|A| \cdot |B|$ not too small

$$\leq \rho \sqrt{|A| \cdot |B|}$$

where $\rho = \max_{i > 1} |\lambda_i|$

Jan 11:

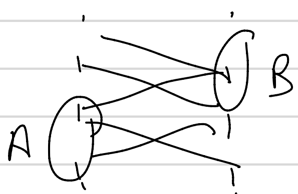
Graph $G=(V,E)$:

$A,B \subset V$, (1) $E(A,B) = \{$ set edges joining A & B

(2) $e(A,B) = |E(A,B)|$

(3) $\Gamma(A) := \{ v \in V \mid v \notin A,\ v \text{ has edge to an element of } A \}$

Bipartite graph



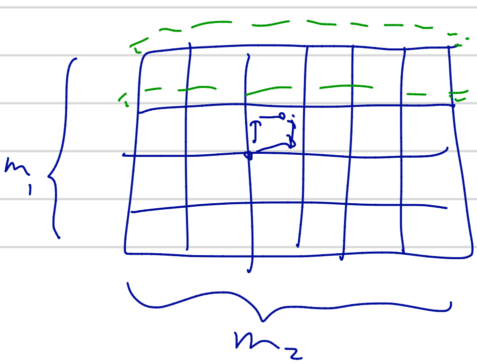
In classical notion of expander (for switching networks)

- given A, want $|\Gamma(A)|$ large
- given A, B want $e(A,B)$ large

Fix d, think of $n \to \infty$, want
- d-regular graph
- d-regular bipartite graph

with good $\begin{cases} \text{edge expansion} \\ \text{vertex expansion} \end{cases}$

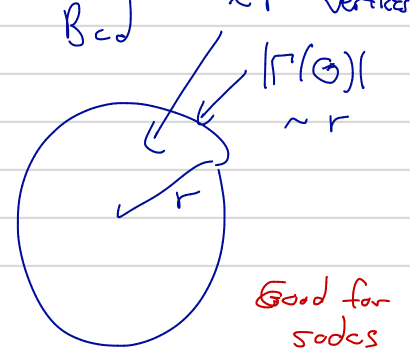
Bad expander: $d=4$



Grid $\mathbb{Z}/m_1\mathbb{Z} \times \mathbb{Z}/m_2\mathbb{Z}$

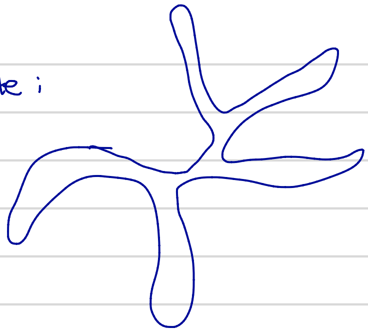
+ wrap around

Bad



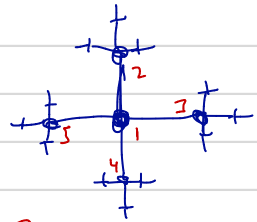
Good for codes

Opposite:



no hotspot in going inside to outside

Say $d=4$



at $\sim O(\sqrt{n})$ labels, see repeat

best locally expander

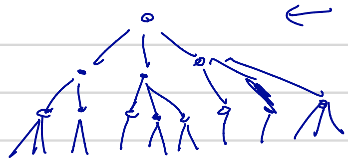
locally tree

$V=\{1,\dots,n\}$

Building a graph that is a local tree is easy.

Girth(graph) := length of shortest cycle

if girth ≥ 7



$\leftarrow v$

$\leftarrow \Gamma(v)$

$\leftarrow \Gamma_2(v) = \Gamma(\{v\}\cup\{\Gamma(v)\})$

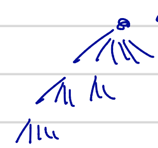
$\Gamma_3(v) \rightarrow$

<u>Open Problem</u>: Given fixed d, large n

$$\text{Girth}\left(\begin{array}{c}\text{any graph } d\text{-reg}\\ \text{on } n\text{-vertices}\end{array}\right) \leq (2+o(1)) \log_{d-1}(n)$$

precise value

Moore bound 1950's



$\leftarrow 1$

$\leftarrow d$

$d(d-1)$

$d(d-1)^2$

$\vdots$

repeat

$> n$

Random graph with corrections of a few bad edges: girth $= (1+o(1)) \log_{d-1}(n)$

LPS-Margulis expanders (bipartite) $\frac{4}{3} \log_{d-1}(n)$ $d = 1+\text{prime}$

Open: Fix any integer $d \geq 3$. Is

$$\text{girth}\begin{pmatrix} \text{All } d\text{-reg graphs} \\ \text{on } n \text{ vertices} \end{pmatrix} \leq (2-\varepsilon)\log_{d-1}(n)$$

for some $\varepsilon > 0$, n suff large.

=

Vertex Expander: $(\alpha, \beta, \gamma, \delta, \varepsilon)$-expander: G, d-regular on n vert

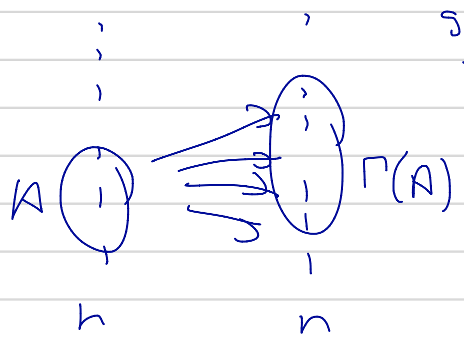
$\forall A \subset V$: if $|A| \leq \alpha n$

$|\Gamma(A)| \geq \beta |A|$

if: $n\gamma \leq |A| \leq n\delta$

then $|\Gamma(A)| \geq \varepsilon |A|$

Vertex Expansion: d-reg (bipartite) graph on n-vertices s.t.



$$|\Gamma(A)| \geq 2|A|$$

for all $A \subset V$ such that

$$|A| \leq \frac{n}{3}$$

You have d fixed, $n \to \infty$, choose d random permutations on $[n] = \{1, \dots, n\}$

- Edge expansion
- Eigenvalue expansion

Jan 14:

Last time: 2 open problems:

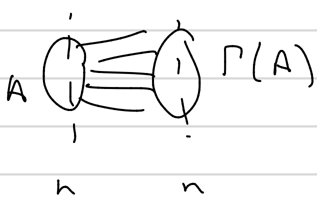
1950's

① Moore bound: $\text{girth}\left(\begin{array}{c}\text{any } d\text{-reg} \\ \text{graph on } n \\ \text{vertices}\end{array}\right) \le 2\log_{d-1} n + o(1)$

Does $2 \rightsquigarrow 2-\varepsilon$ hold?

more is known

② Construct bipartite graph d-regular with smallest d, $n \to \infty$



s.t. for all $A \subset$ left vertices, $|A| \le \frac{n}{3}$

$$|\Gamma(A)| \ge 2|A|.$$

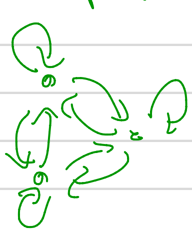
Both fundamentally connected to eigenvalues of Adjacency(G).

Today: ① Examples of Graphs & Eigenvalues

② More on "Expansion"

Classical Examples of Good/Bad Connectivity in Graphs:

Examples:



$$n\left\{\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \vdots & & & & \\ 1 & & \cdots & & 1 \end{bmatrix}}_{n}\right.$$

λ's: n mult. 1

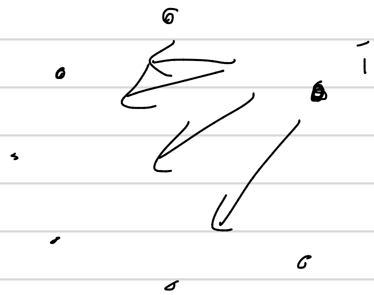
0 mult $n-1$

$$\lambda_1 = n \ge \underbrace{\lambda_2 \ge \cdots \ge \lambda_n}_{0}$$

☺ Connecti

☹ degree $d=n$

(2) Graph: $V = \mathbb{Z}/p\mathbb{Z} = \{0, \dots, p-1\}$



For $i \in V$, edge

$i \longrightarrow i+j$ where j is a quadratic residue,

$i \longrightarrow i + a^2 \bmod p$,

$a = 1, \dots, p-1$

$a^2 = (-a)^2$,

Def: We say that j is a quadratic residue mod p if $j \not\equiv 0 \pmod p$ and $j \equiv a^2 \pmod p$ for some a.

e.g. $p = 11$, $1^2 \equiv 1 \pmod{11}$, $2^2 \equiv 4 \pmod{11}$, $3^2 \equiv 9 \pmod{11}$,

$4^2 \equiv 5 \pmod{11}$, $5^2 \equiv 3 \pmod{11}$

$6^2 \equiv (-5)^2$ done, $7^2 \equiv (-4)^2$ done, ...

Quadratic residues mod 11: $1, 3, 4, 5, 9$ mod 11

" non-residues " " : $2, 6, 7, 8, 10$ mod 11

Fact: res $\cdot$ res $\equiv$ res

(non-res) $\cdot$ res $\equiv$ non-res

(non-res)(non-res) $\equiv$ res

$2^0, 2^1, 2^2, \dots, 2^{10}$ get all non-zero mod 11

$1, 2, 4, 8, 5, \dots, 6$

Homework: ① If p is prime, $i \neq 0$, $x^2 \equiv i \pmod p$ has either 2 solutions or no solutions

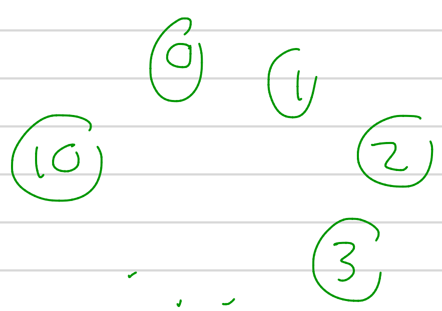
② There are $\frac{p-1}{2}$ values of i s.t. $x^2 \equiv i \pmod p$ has 2 solutions [such i are called Quadratic Residues]

③

$$\text{res} \cdot \text{res} \equiv \text{res}$$

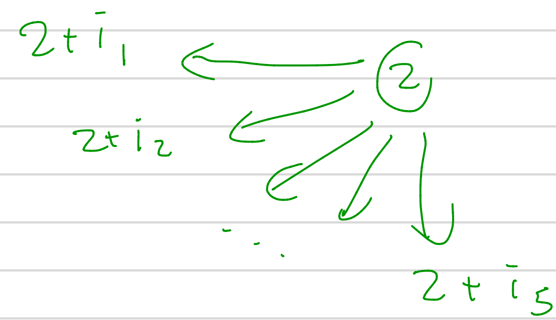
$$(\text{non-res}) \cdot \text{res} \equiv \text{non-res}$$

$$(\text{non-res})(\text{non-res}) \equiv \text{res}$$



$p = 11$

$\leftarrow V = \mathbb{Z}/p\mathbb{Z} = \{0, 1, \dots, p-1\}$



$\{i_1, i_2, \dots, i_5\}$ are quad residues, i.e.

$$\{1, 3, 4, 5, 9\}$$

Claim: This graph is d-regular, with $d = \frac{p-1}{2}$ (here 5)

$$d = \lambda_1 \geq \underbrace{\lambda_2 \geq \dots \geq \lambda_p}_{\text{small}}$$

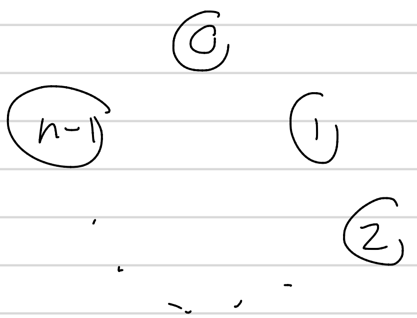
☺ good "expansion"

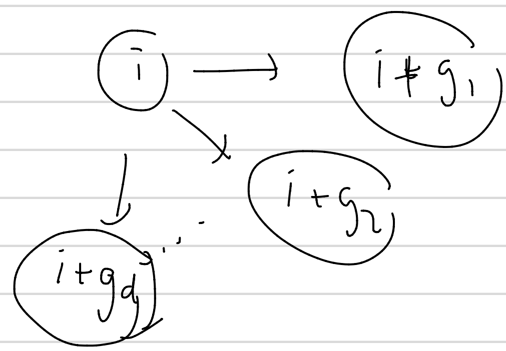
☹ $d = \frac{p-1}{2} \approx \frac{n}{2}$

Fact: Say:

$$\text{Graph} = \text{Cayley}\left(\mathbb{Z}/n\mathbb{Z},\ \{\overbrace{g_1, \ldots, g_d}^{\text{"generators"}}\}\right)$$

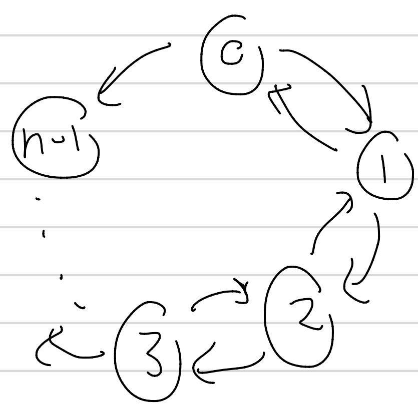
Vertices: $\mathbb{Z}/n\mathbb{Z}$ $\qquad$ $g_1, \ldots, g_d \in \mathbb{Z}/n\mathbb{Z}$



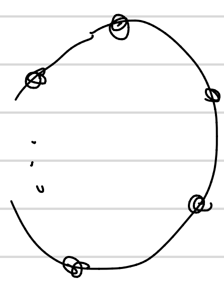


Example:

$$\text{Cayley}\left(\mathbb{Z}/n\mathbb{Z},\ \{1, -1\}\right)$$



also



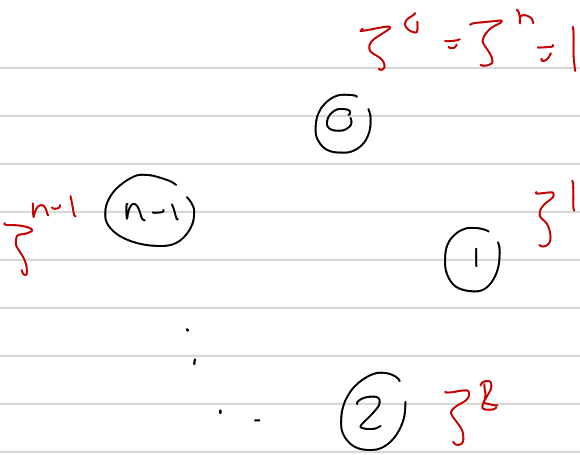
cycle length n

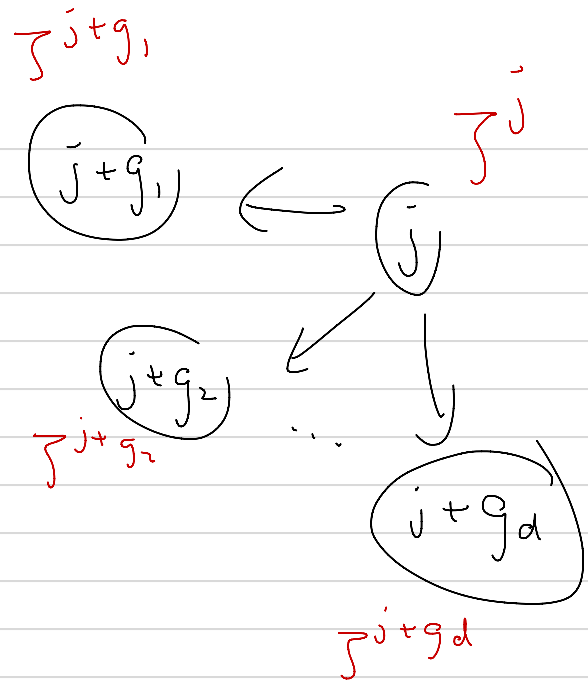
bad expander

If $\zeta^n = 1$, i.e. ζ is an n-root of unity, i.e. $\zeta = \left(e^{\frac{2\pi i}{n}}\right)^k$

Then $f: V \to \mathbb{C}$, $f(j) = \zeta^j$ is an eigenfunction

$\zeta^n = 1$;





$$\left(\text{Adj op on } f\right)(j)$$

$$= \sum_{j' \sim j} f(j')$$

if $f = \begin{bmatrix} \zeta^0 \\ \zeta^1 \\ \vdots \\ \zeta^{n-1} \end{bmatrix}$

$$= \zeta^j\left(\zeta^{g_1} + \zeta^{g_2} + \dots + \zeta^{g_d}\right)$$

So $\text{Adj} \begin{bmatrix} \zeta^0 \\ \zeta^1 \\ \vdots \\ \vdots \end{bmatrix} = \left(\qquad \right) \begin{bmatrix} \zeta^0 \\ \zeta^1 \\ \vdots \\ \vdots \end{bmatrix}$

Jan 16:

Multiply: $p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$

$q(x) = b_0 + \quad \dots \quad + b_n x^n$

Naïve way to compute $p(x)q(x)$ is $O(n^2)$ $\begin{cases} \text{FLOPs} \\ \text{Time} \\ \vdots \end{cases}$

Imagine you can compute $p(x), q(x)$ at 2^m-th roots of unity $2^m \geq 2n$ (or any ~~to~~ points $\geq 2n$)

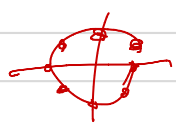
$$\boxed{\text{can find } \begin{array}{l} p(y_1), \dots, p(y_{2n}) \\ q(y_1), \dots, q(y_{2n}) \end{array} \text{ "quickly"}}$$

"Fast discrete Fourier transform" for certain $y_1, \dots, y_{2n}$ roots of unity, $\{\zeta^m = 1\}$ same m

can compute $(pq)(y_1), \dots, (pq)(y_{2n})$ in $2n$ mults

$$\boxed{\text{Say given deg } 2n \text{ poly } r = r(x), \text{ and given } r(y_1), \dots, r(y_{2n}) \text{ you can find } r(x) \text{ quickly}}$$

Image: $\zeta^8 = 1$



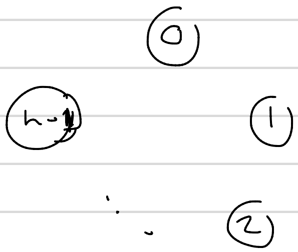
ζ, ζ^2, ζ^4

$\zeta = -\zeta^5$

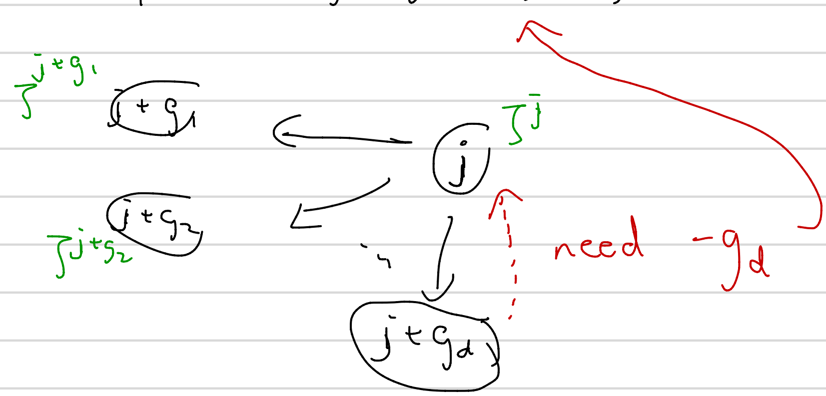
$e^{2\pi i x}$

For us:

$\mathbb{Z}/n\mathbb{Z}$



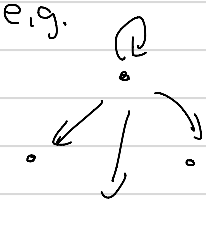
$G = \text{Cayley}(\mathbb{Z}/n\mathbb{Z}, \{g_1, \dots, g_d\})$:



Claim: For any ζ s.t. $\zeta^n = 1$, eigenpair

$$\text{Adj}_G \begin{bmatrix} \zeta^0 \\ \zeta^1 \\ \vdots \\ \zeta^{n-1} \end{bmatrix} = \left(\zeta^{g_1} + \dots + \zeta^{g_d}\right) \begin{bmatrix} \zeta^0 \\ \vdots \\ \zeta^{n-1} \end{bmatrix}$$

=

e.g. 

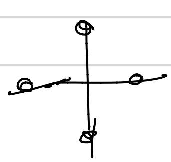
complete digraph on 0, 1, 2, 3 (4 vertices)

$$\text{Adj} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

$\lambda = 4, 0, 0, 0$

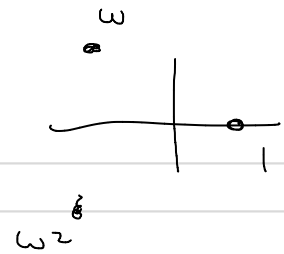
rank 1

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} i^0 \\ i^1 \\ i^2 \\ i^3 \end{bmatrix} = \begin{bmatrix} \text{all } 1\text{'s} \end{bmatrix} \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix}$$



Similarly $\quad \lambda = 3, 0, 0$

$\omega = e^{2\pi i/3}$



$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix} = 0$$

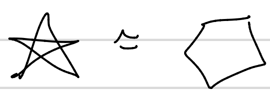
$$'' \quad \begin{bmatrix} 1 \\ \omega^2 \\ \omega \end{bmatrix} = 0$$

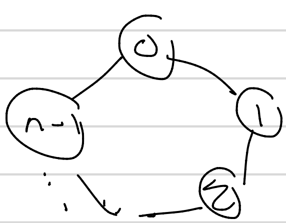
$$'' \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathrm{Cayley}\Big(\mathbb{Z}/n\mathbb{Z},\ \underset{\{0,1,\dots,n-1\}}{\overset{\mathbb{Z}/n\mathbb{Z}}{\shortparallel}}\Big)$$

Worst Connected Graph Regular in term expansion:

$\mathrm{Cayley}(\mathbb{Z}/n\mathbb{Z}, \{1,-1\})$





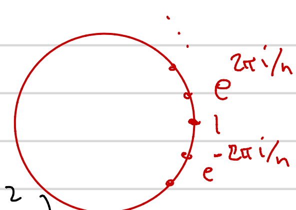
cycle of length n

Formula says: take any ζ with $\zeta^n = 1$, $\quad \lambda = \lambda_\zeta = \zeta + \zeta^{-1}$

λ_1 = largest eigenvalue $= 2$, $\quad \lambda_2$ = next largest =

$$e^{2\pi i/n} + e^{-2\pi i/n}$$

$$= 2\cos\left(\frac{2\pi}{n}\right) = 2\left(1 - \left(\frac{2\pi}{n}\right)^2 + \cdots\right) = 2 - \mathrm{order}\left(\frac{1}{n^2}\right)$$

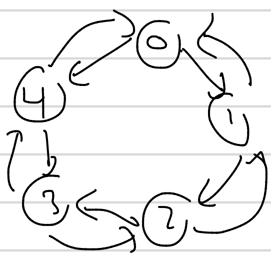


p prime $\quad$ Cayley$\left(\mathbb{Z}/p\mathbb{Z}, \{\text{all quadratic residues}\} \right)$

OR

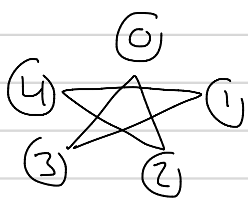
" " " " non-residues

$p = 5$ $\quad$ $1^2 \equiv 1, 2^2 \equiv 4, 3^2 \equiv 4, 4^2 \equiv 1 \mod 5$

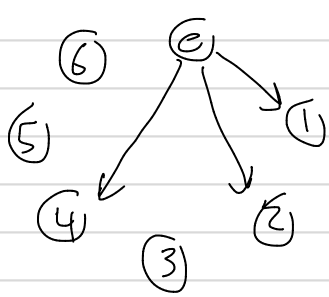


quad residues $\quad 1, -1$

non-residues $\quad 2, -2$



$p = 7$ $\quad$ $1^2 \equiv 1, 2^2 \equiv 4, 3^2 \equiv 2, \quad 2, 4, 1$



-1 is not a quad residue

Fact: -1 is quad residue mod p, $\overset{p \text{ odd prime}}{}$ iff $p \equiv 1 \pmod 4$

Degree: $\dfrac{p-1}{2}$

Question: If $\zeta^p = 1$, what is $\displaystyle\sum_{\substack{r \in \text{quad} \\ \text{res mod } p}} \zeta^r$ for $p \equiv 1 \pmod 4$

Homework: p prime (maybe $p \equiv 1 \pmod 4$)

What is

$$\lambda_\zeta = \frac{1}{2} \sum_{\substack{a \not\equiv 0 \\ (\mathrm{mod}\ p)}} \zeta^{a^2}, \quad \text{for } \zeta \neq 1 \quad ?$$

$\lambda_1 = \frac{p-1}{2}$. $\zeta \neq 1$ but $\zeta^p = 1$

Hint

$$\left(\sum_{a \not\equiv 0} \zeta^{a^2} \right)^2 = \sum_{a \neq 0} \zeta^{a^2} \overline{\sum_{b \neq 0} \zeta^{b^2}}$$

$$= \sum \zeta^{a^2 - b^2}$$

$$= \sum_{a, b \neq 0} \zeta^{(a+b)(a-b)}$$

$$= \sum_{\{\alpha, \beta\}} \zeta^{\alpha \beta}$$

Jan 18:

$$\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$$ d-reg graph, n vertices:

$d = n$ big ☹

↑ row sums = col sums

$$\lambda_1 = n = d \geq \lambda_2 \geq \cdots \geq \lambda_n \geq -d$$

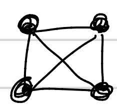
here $\lambda_2, \ldots, \lambda_n = 0$ ☺

=

Varient

$$\begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ & 0 & & 1\text{'s} & \\ & & 0 & & \\ 1\text{'s} & & & 0 & \\ & & & & 0 \cdots 0 \end{bmatrix}$$

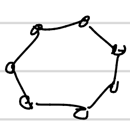
complete graph $= A_{\text{complete digraph}} - I$



$$\lambda_1 = d = n-1, \quad \lambda_2 = \cdots = \lambda_n = -1$$

=

Cycle



$$d = 2, \quad \lambda\text{'s}: 2\left(\cos \frac{2\pi}{n} \cdot m\right) \quad m = 0, \ldots, n-1$$

$$\lambda_1 = d = 2 \geq \lambda_2 = 2\cos\frac{2\pi}{n} = 2 - \underbrace{\text{order}\left(\frac{1}{n^2}\right)}_{\frac{2\cdot(2\pi)^2}{2!}\frac{1}{n^2} + O\left(\frac{1}{n^4}\right)}$$

=

Given d-regular graph on n vertices, let

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

$\lambda_i = \lambda_i(G)$ are the eigenvalues of A = Adjacency matrix of G

Thm: Fix integer $d \geq 3$. Any graph on n

=

$$\lambda_2 \geq 2\sqrt{d-1}\left(1 - \left(\frac{1}{\log_{d-1}(n)}\right)^2\right.$$

Fix $d \geq 3$. For a graph G d-regular on n vertices, we have

$$\lambda_2(G) \geq 2\sqrt{d-1}\left(1 - O\left(\frac{1}{\log_{d-1} n}\right)^2\right)$$

[$\log_{d-1}$ familiar from girth/Moore bound]

i.e.

$$2\sqrt{d-1}\,(1 + o(1))$$

Thm: Fix $d \geq 3$. Then for any $\varepsilon > 0$,

$\underbrace{\text{for "most" } d\text{-regular graphs}}$ on n vertices, as $n \to \infty$

$\underbrace{\text{with high probability}}_{\text{WHP}}$ for a random graph - - - - - -

$$\boxed{\begin{aligned} \lambda_2 &\leq 2\sqrt{d-1} + \varepsilon \\ -(2\sqrt{d-1} + \varepsilon) &\leq \lambda_n \end{aligned}}$$

$(\lambda_2 \leq d)$

$(-d \leq \lambda_n)$

$(d = 2,\ 2\sqrt{d-1} = 2)$

$(d = 3,\ 2\sqrt{d-1} = 2\sqrt{2} < 3)$

$\vdots$

$$\rho(G) := \max_{i \geq 2} |\lambda_i| = \max(\lambda_2, -\lambda_n)$$

$$\rho(G) \leq 2\sqrt{d-1} + \varepsilon$$

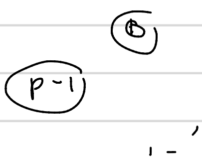
Upshot: G: $\rho(G) := \max_{i \geq 2} \left| \lambda_i(\text{Adjacency}_G) \right|$

① $$\left| e(U,W) - \frac{d}{n}|U|\,|W| \right| \leq \rho \sqrt{\frac{|U|(n-|U|)}{n}} \sqrt{\frac{|W|(n-|W|)}{n}}$$

$$\leq \rho \sqrt{|U|}\sqrt{|W|}$$

"expander mixing lemma"

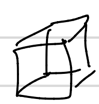
② If $p \equiv 1 \pmod 4$

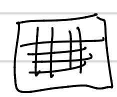


$\text{Cayley}(\mathbb{Z}/p\mathbb{Z}, \{\text{quad residues}\})$

$n = p$, $d = \frac{p-1}{2}$, $\rho \approx$ better than $2\sqrt{d-1}$

③ A few more examples... General tools ...





To prove ①:

$$\mathrm{Adj}_G = \underbrace{\frac{d}{n}\begin{bmatrix}1 & \cdots & 1\\ \vdots & \ddots & \vdots\\ 1 & \cdots & 1\end{bmatrix}} + \overset{\text{error term}}{\mathcal{E}}$$

$$= d\Big(\underbrace{\mathrm{proj}_{\vec{1}}}_{\left(\frac{\vec{1}}{\sqrt{n}}\right)\left(\frac{\vec{1}}{\sqrt{n}}\right)^T}\Big) + \underbrace{\mathrm{Adj}_G\Big|_{\vec{1}^\perp}}_{\text{norm in } L^2 \text{ op sense } = \rho}$$

$=$

$$A_{sym} = \sum_{i=1}^{n} \lambda_i \underbrace{\vec{u}_i \vec{u}_i^T}$$

for orthonormal eigenbasis $\vec{u}_1, \ldots, \vec{u}_n$

$$= \sum_{i=1}^{n} \lambda_i \overbrace{\mathrm{proj}_{u_i}}$$

$$= \lambda_1 u_1 u_1^T + \sum_{i \geq 2} \lambda_i \mathrm{proj}_{u_i}$$

$=$

e.g.

$$A = \begin{bmatrix}1 & \cdots & 1\\ \vdots & \ddots & \vdots\\ 1 & \cdots & 1\end{bmatrix} - I = \mathrm{Adj}_{\text{complete graph}}$$

$d = \lambda = n-1, -1, -1, \ldots, -1$

$$= \frac{d}{n}\begin{bmatrix}1 & \cdots & 1\\ \vdots & \ddots & \vdots\\ 1 & \cdots & 1\end{bmatrix} + (-1)\,\mathrm{proj}_{\vec{1}^\perp}$$

Jan 21:

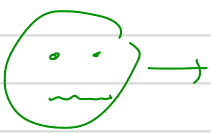
$$D = \begin{bmatrix} d_1 & & 0\text{'s} \\ & d_2 & \\ 0\text{'s} & & \ddots \; d_n \end{bmatrix} = \underbrace{\begin{bmatrix} d_1 & & \\ & 0 & \\ & & 0 \; \ddots \end{bmatrix}}_{D|_{\vec{e}_1}} + \underbrace{\begin{bmatrix} 0 & & & \\ & d_2 & & \\ & & d_3 \; \ddots & \\ & & & d_n \end{bmatrix}}_{D|_{\underbrace{\text{span}(\vec{e}_2, \dots, \vec{e}_n)}_{\{e_1\}^\perp}}} = \text{etc.}$$

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \dots, \vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$D\vec{e}_i = d_i \vec{e}_i$$

$$D = \underbrace{\begin{bmatrix} d_1 & & & & \\ & d_2 & & & \\ & & 0 & & \\ & & & d_4 & \\ & & & & 0 \; \ddots \end{bmatrix}}_{\text{span}(\vec{e}_1, \vec{e}_2, \vec{e}_4)} + \underbrace{\begin{bmatrix} 0 & & & & & \\ & 0 & & & & \\ & & d_3 & & & \\ & & & 0 & & \\ & & & & d_5 & \\ & & & & & d_6 \; \ddots \end{bmatrix}}_{\text{span}(\vec{e}_3, \vec{e}_5, \dots, \vec{e}_n)}$$

Really
$d_1 = \lambda_1$
$d_2 = \lambda_2$
$\vdots$
in this case



Why works: $D|_{\underbrace{\text{span}(\vec{e}_1, \vec{e}_2, \vec{e}_4)}_{S_1}}$ image $\subset$ $\underbrace{\text{span}(\vec{e}_1, \vec{e}_2, \vec{e}_4)}_{S_2}$

$D : S_1' \to S_1'$

$D : S_2 \to S_2'$

$$\mathbb{R}^n = \text{span}(S_1', S_2') = S_1' \oplus S_2'$$

Similarly for $D = \begin{bmatrix} d_1 & & \\ & 0 & \\ & & \ddots \end{bmatrix} + \begin{bmatrix} 0 & & \\ & d_2 & \\ & & 0 \; \ddots \end{bmatrix} + \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & d_3 & \\ & & & 0 \; \ddots \end{bmatrix} + \dots$

RETURN TO THIS LATER

Difficult open problems:

Question: For fixed d and n large ($n < d^2$ or so uninteresting) does there exist a d-reg graph on n vertices with $\rho \le 2\sqrt{d-1}$? Can you "construct" one?

Question: For fixed d and n large, for how large a $k = k(d)$ can you "construct" a graph such that if $|A|, |B| \ge \frac{n}{k}$, $\# E(A,B) \ge 1$?

Question: Fix any $d \ge 3$. Is it true that as $n \to \infty$, for all d-regular graphs, G, on n vertices,

$$\text{girth}(G) \le (2-\varepsilon)\log_{d-1} n$$

for some $\varepsilon > 0$.

Graph: $U \subset \vec{V} \leftarrow$ vertices

$$\chi_{\vec{U}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix} = \left\{ \begin{array}{l} \text{the } v^{th} \text{ component of} \\ U \text{ is } 1 \text{ if } v \in \vec{U} \\ \text{otherwise } 0 \end{array} \right.$$

$\mathbb{R}^{\vec{V}}$

$\mathbb{R}^{n}$, where $n = |V|$

=

Rem: $\vec{U}, \vec{W} \subset \vec{V} \leftarrow$ vertices of a graph

$$\chi_U^T \,(\text{Adj of } G)\, \chi_W = [1\,0\,1\,0\ldots] \begin{bmatrix} \text{Adj} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ \vdots \end{bmatrix}$$

$$= e(U, W)$$

=

But: $\mathbb{R}^V = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \oplus \left\{ \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \right\}^{\perp}$

$$\text{proj}_{\vec{1}}(\chi_U) = \left((\chi_U) \cdot \begin{pmatrix} 1/\sqrt{n} \\ \vdots \\ 1/\sqrt{n} \end{pmatrix} \right) \begin{bmatrix} 1/\sqrt{n} \\ \vdots \\ 1/\sqrt{n} \end{bmatrix} = \frac{|\vec{U}|}{\sqrt{n}} \left(\vec{1}/\sqrt{n} \right)$$

$$= \frac{|U|}{n} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \frac{|U|}{n} \vec{1}$$

$$\text{proj}_{\{\vec{1}\}^{\perp}}(\chi_U) = \chi_U - \text{proj}_{\vec{1}} \chi_U = \chi_U - \frac{|U|}{n} \vec{1}$$

$$\Big| \underbrace{\qquad\qquad}_{} \Big| = \text{magic} \sqrt{\frac{|U|\,(n - |U|)}{n}}$$

is this intuitive $\left\{ \begin{array}{l} U = \emptyset \checkmark \\ U = \vec{V} \end{array} \right.$

General Facts:

If A is symmetric,

$S_1 =$ span some eigenvect in ON eigenbasis

$S_2 = S_1^{\perp}$

then

$A|_{S_1} : S_1 \to S_1$

$A|_{S_2} : S_2 \to S_2$

both symmetric

$$A = \begin{bmatrix} \lambda_1 & & 0's \\ & \lambda_2 & \\ 0's & & \ddots \ \lambda_n \end{bmatrix}$$

wrt a "nice" orthonormal basis

Fact: Say A sym, eigs $\lambda_1 \geq \ldots \geq \lambda_n$

Then

$$\|A\|_{L^2\text{-operator}} = \max_{\vec{v} \neq 0} \frac{\|A\vec{v}\|_2}{\|\vec{v}\|_2} = \max_{i=1,\ldots,n} |\lambda_i|$$

Pf: Say that

Case 1: $A = \begin{bmatrix} \lambda_1 & & 0's \\ & \ddots & \\ 0's & & \lambda_n \end{bmatrix}$, $\vec{v} = \sum c_i \vec{e}_i$

by scaling $\vec{v}$: can assume $\sum c_i^2 = 1$

Then

$$A\vec{v} = A\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \lambda_1 c_1 \\ \vdots \\ \lambda_n c_n \end{bmatrix}, \text{ norm}^2: \sum \lambda_i^2 c_i^2$$

if $\vec{v} = \vec{e}_j$, $\text{norm}^2 = \lambda_j^2$