Sept. 20

$\begin{cases} \text{Finite} \\ \text{Infinite} \to \text{Countably Infinite} \\ \phantom{\text{Infinite}} \to \text{Uncountable} \end{cases}$

Finite "<" Countably Infinite "<" Uncountable

e.g. $\{a,b,c\}$ $\mathbb{N} = \{1,2,3,\ldots\}$ $\mathbb{R}$ = real numbers

$\mathbb{Z} = \{\ldots,-1,0,1,2,\ldots\}$

A^* strings over an alphabet

Languages over an alphabet

Definition: We say a set S is countable if there is a surjection $f : \mathbb{N} \to S$, i.e. f is onto, i.e. for each $s \in S$ there is an $n \in \mathbb{N}$ s.t. $f(n) = s$.

E.g. $\mathbb{Z} = \{\ldots,-1,0,1,2\}$

$= \{0, 1, -1, 2, -2, 3, -3, \ldots\}$

f: $f(1)=0$, $f(2)=1$, $f(3)=-1$, $f(4)=2$, $\ldots$

$f : \mathbb{N} \to \mathbb{Z}$ is onto

Rational numbers: $\mathbb{Q}$ = Reals can be written as a/b, $a,b \in \mathbb{Z}$

a/b b \ a	0,	1,	-1,	2,	-2,	3,	-3
0	$\cancel{0/0}$	$\cancel{1/0}$	$\cancel{-1/0}$	$\cancel{2/0}$	$-2/0$	-	-
1	$\cancel{0/1}$	$1/1$	$-1/1$	$2/1$	-	-	
-1	$0/-1$	$1/-1$	$-1/-1$	$2/-1$	-	-	-
2	-		-				
-2				-			
,	-		-				
,						$-3/5$	-

Creates a list of all rational numbers

You can make a list

$q_1, q_2, q_3, \ldots$ of all rational numbers

i.e. $f: \mathbb{N} \to \mathbb{Q}$

So $\mathbb{N}, \mathbb{Q}, \mathbb{Z}$ are infinite, but countable.

Which sets are uncountable? Reals $\mathbb{R}$, any open real interval,

To show a set is uncountable is not necessarily easy...

Theorem: If S is a set, then there is no surjection $f: S \to Power(S)$. !!

[$Power(S)$ = set of all subsets of S, sometimes written 2^S]

e.g. $S = \{a, b\}$, $Power(S) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$

$|S| = |\{a,b\}| = 2$, $|Power(S)| = 2^{|S|} = 2^2 = 4$

if $|S| = n$, $|Power(S)| = 2^n$

Claim: $n < 2^n$.

Theorem: Given $f: S \to Power(S)$, there is a T s.t. $T \subset S$ but $T \neq f(s)$ for any $s \in S$.

Pf: Given $f: S \to Power(S)$, let

$$T = \{ s \in S \text{ such that } s \notin f(s) \}$$

negation

Pause for an example

$f: \{1,2,3\} \to Power(\{1,2,3\})$

$1 \mapsto \{1,2\} = f(1)$

$2 \mapsto \{1,3\} = f(2)$

$3 \mapsto \{1,2\} = f(3)$

$T = \{ \ ? \ \}$

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$1 \in f(1)$ so "1 is not s.t. $1 \notin f(1)$"

$1 \notin f(1)$ is false

$2 \notin f(2)$ is true

$3 \notin f(3)$ is true

So $T = \{2, 3\}$.

~~Can~~ Can $f(1) = T$? Hope NO

$f(2) = T$? " "

$f(3) = T$? " "

$T = \{ s \in S \text{ s.t. } s \notin f(s) \}$

~~If $f(1) = T$ then $1 \notin f(1)$ $T = \{2, 3\}$~~

~~If $f(1) = T$ $1 \notin f(1) = T = \{2, 3\}$~~

If $f(1) = T$ <
- $1 \in f(1) = T$ then, by def T, $1 \notin f(1)$
- $1 \notin f(1) = T$, then, by def T, $1 \in f(1)$

Formal Proof: Given $f : S \to \text{Power}(S)$, let $\boxed{T = \{ s \in S \mid s \notin f(s) \}}$. Then T is not in the image of f, since if $f(t) = T$ some $t \in S$:

if $t \in f(t) \Longrightarrow t \notin f(t)$

if $t \notin f(t) \Longrightarrow t$ does not satisfy $s \notin f(s)$ $t = s$, $t \in f(t)$

Notation: $A \subset B \iff A$ is a subset of B

$A \subseteq B$ $(A \subsetneq B)$

Example: If $T = \emptyset = \{s \in S \mid s \notin f(s)\}$

then no s satisfies $s \notin f(s)$

$\Rightarrow$ all $s \in f(s)$

$1 \in f(1),\ 2 \in f(2),\ 3 \in f(3), \ldots \quad s \in f(s)$

Can $\emptyset$ appear   no

so $\emptyset \neq$ any of the $f(s)$'s

So... $\mathbb{N} \to \text{Power}(\mathbb{N})$ can't get a surjection

so $\text{Power}(\mathbb{N})$ is <u>uncountable</u>

If A is an alphabet, A is a nonempty finite set,

A^* is countable:

$\{a, b\} = A$, $\quad A^k = \{\text{words of length } k\}$

$A^* = A^0 \cup A^1 \cup A^2 \cup \ldots$

$A^0 = \{\varepsilon\},\ A^1 = \{a, b\},\ A^2 = \{aa, ab, ba, bb\}, \ldots$

So the list $\varepsilon, a, b, aa, ab, ba, bb, \ldots$

hits every elt of A^*

But $\{\text{languages over } A\} = \text{Power}(A^*)$ uncountable