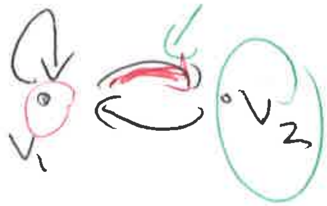
Warmups to Ch 1,2
- Walk Counts
- Non-"walk counts"
- Self-referencing; sets; uncountability, etc.



walks length n from v_1 to v_1 = Fib_{n+1}

" " " " v_2 to v_2 =

" " $n-1$ " v_1 to v_1 = Fib_n

walk from v_1 to v_2

$v_1, \ldots\ldots,$ $v_1 \to v_2$

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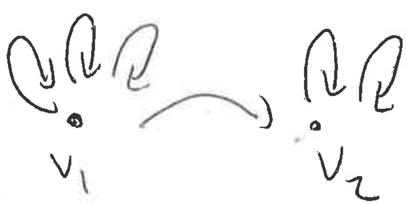
walks length n from v_1 to $\{v_1, v_2\}$ = $F_{n+1} + F_n = F_{n+2}$



walks length n from
- $v_1 \to v_1$: 3^n
- $v_2 \to v_2$: 2^n

- $v_1 \to v_2$:

$$v_1 \xrightarrow{1} \underbrace{v_2 \xrightarrow{2} \overset{2}{-} \overset{2}{-} \overset{2}{-} \to v_2}_{n-1} \qquad 2^{n-1}$$

$$v_1 \xrightarrow{3} v_1 \xrightarrow{1} \underbrace{v_2 \xrightarrow{2} - \; v_2}_{n-2} \qquad 3 \cdot 2^{n-2}$$

$$\vdots$$

$$\underbrace{v_1 \xrightarrow{3} v_1 \xrightarrow{3} - \; - \; v_1}_{n-1} \xrightarrow{1} v_2 \qquad 3^{n-1}$$

$$2^{n-1} + 3 \cdot 2^{n-2} + \ldots + 3^{n-1} = 3^{n-1}\left(\frac{2^{n-1}}{3^{n-1}} + \ldots + \frac{2^2}{3^2} + \frac{2}{3} + 1\right)$$

$$= 3^{n-1}\left(\frac{1-(2/3)^n}{1-(2/3)}\right) = 3\left(3^{n-1} - 3^{n-1}\frac{2^n}{3^n}\right)$$

$$= 3^n - 2^n$$

$v_1 \to v_1 \quad 3^n$

$v_1 \to v_2 \quad 3^n - 2^n$

$v_2 \to v_2 \quad 2^n$

walks length n from $\{v_1, v_2\}$ to $\{v_1, v_2\}$ $= 2 \cdot 3^n$

<u>Def</u>: If G is digraph $G = (V, E, h, t)$ and $V_1, V_2 \subset V$

we define $f(n) = \text{Walks}(n, V_1, V_2, G)$

$=$ # walks length n starting in any vertex in V_1 ending in any vertex of V_2 in G

Say that $f(n)$ is a <u>walk-counting function</u> if $f(n): \mathbb{N} \to \mathbb{Z}_{\geq 0}$ given by $\text{Walks}(n, V_1, V_2, G)$ for some digraph G, $V_1, V_2 \subset$ vertices of (G).

We know: Fib_{n+1}, Fib_n, Fib_{n+2}, 2^n, 3^n, $3^n - 2^n$, ... are walk-counting functions.

Bonus questions on homework:

Questions: Say $f_1(n)$, $f_2(n)$ are walk counting functions.

Is $f_1(n) + f_2(n) = f_{new}(n)$ also a walk counting function?

(E.g. is $f(n) = Fib_{n+2} + 2^n$ a walk counting function?)

Answer yes:

informally: if Fib_{n+2} counts  from $\{v_1\}$ to $\{v_1, v_2\}$

if 2^n counts  from w_1 to w_1

$\sqcup$ new digraph  "stitch together"

walks length n start in $\{v_1, w_1\}$ end $\{v_1, v_2, w_1\}$

$= Fib_{n+2} + 2^n$

Let $f_1(n)$ be walk-counting $\mathrm{Walks}(n, V_1, V_2, G)$ (V, E, h, t)

$f_2(n)$ " $\mathrm{Walks}(n, W_1, W_2, H)$ (W, F, h', t')

Then form new digraph $G \amalg H$: vertices $V \amalg W$ disjoint union

edges $E \amalg F$

head

tails

What functions $f(n): \mathbb{N} \to \mathbb{Z}_{\geq 0}$ can't be walk counting functions?

Thm (from linear algebra):

If $f(n)$ has asymptotic ratio $\rho > 0$, then

- if $f(n) = o(\rho^n)$
- if ρ is a rational that is not an integer
- ...
- ...

then $f(n)$ is not a walk-counting function.

Ex: Say $f(n) = 2^n/\sqrt{n} + o(2^n/\sqrt{n})$ leading term

then $\rho(f) = \lim_{n\to\infty} \frac{f(n+1)}{f(n)} = \lim_{n\to\infty} \frac{2^{n+1}/\sqrt{n+1}}{2^n/\sqrt{n}}$

$$= \lim_{n\to\infty} 2\frac{\sqrt{n}}{\sqrt{n+1}} = \lim_{n\to\infty} 2\frac{1}{\sqrt{1+1/n}} = 2$$

Since $f(n) = 2^n/\sqrt{n} = o(2^n) \Rightarrow f$ is not walk-counting

e.g. $f(n) = 3^n + 2^n = 3^n + o(3^n)$

$\rho(f) = 3$, but $f(n) = 3^n + 2^n$ is not $o(\rho^n) = o(3^n)$

e.g. $f(n) = F_{n+2}$?? $\rho(F_{n+2}) = \frac{1+\sqrt{5}}{2}$. $F_{n+2} \sim \left(\frac{1+\sqrt{5}}{2}\right)^{n+2} \frac{1}{\sqrt{5}}$