Question 1

First suppose $S \in S$. Then by the definition of S, since $S \in S$ we have that $S \notin S$. This is a clear contradiction.

Now suppose $S \notin S$. By the definition of S then, S belongs in S, contradicting that S is not in S.

I think no "reasonable" set theory can allow S to exist, since no matter whether S contains itself or not there is some logical inconsitency.

Question 2

Let $S = \{0, 1\}$. Note the DFAs are included at the end of the document.

(b) Regular expressions describing $\{w|w$ containing three 1's$\}$.

$S * 1S * 1S * 1S*$, or $0 * 10 * 10 * 1S*$, or $S * 10 * 10 * 1S*$.

One incorrect answer which was submitted is $S * 111S$. This requires three 1s to be consecutive, and thus is too restrictive.

(f) Regular expressions describing $\{w|w$ doesnt contain the substring 110$\}$.

$0 * \cup [(0 * 10) * 1*$, or $0 * (100*) * 1*$, or $0 * (10 \cup 0) * 1*$, or $(10 \cup 0) * 1*$. Incorrect answers that were submitted were: $0 * (10*) * 1*$: note that the middle $(10*)*$ matches 1110, for example. $(0 * 1) * 0 * 1*$: note that the string 110 matches this expression.

(l) Regular expression describing $\{w|w$ contains an even number of 0s, or exactly two 1's$\}$.
$(1 * 01 * 0) * 1 * \cup 0 * 10 * 10*$

Question 3

Let R' be the regular expression of problem 3. It is important to include "both directions" of the explanation that $L(R')$ describes the set of strings with an even number of 0s and an odd number of 1s. (Several solutions only presented the direction that shows $L(R')$ is a subset of L, where L is language of string with an odd number of 1s and an even number of 0s, but omitted to show that L is a subset of $L(R')$.

$L(R')$ **is a subset of** L : Suppose that w is in $L(R')$. We show that w must have an even number of 0s and an odd number of 1s. Note that w must be the concatenation of three strings, say $w = xyz$, where both x and z are in $L(R)$ and y is in $L(1 \cup 01(11) * 0)$. Hence, both x and z must have an even number of 0s and y must have either zero or two 0s, (depending on whether y is in $L(1)$ or y is in $L(01(11) * 0)$). Hence since all of x, y, and z have an even number of 0s, so must w. Also, both x and z must have an even number of 1s, but y must have an odd number of 1s. Since two even numbers plus one odd number is an odd number, clearly w must have an odd number of 1s.

L **is a subset of** $L(R')$: (This is the harder direction, kudos to those of you who understood how to proceed with this one.) Suppose that w has an even number of 0s and an odd number of 1s. We show that w is in $L(R')$. Let x be the longest prefix of w that has an even number of 0s and an even number of 1s. Note that x may be the empty string but x cannot be the whole string w; that is, x is a proper prefix of w. We now consider two cases.

1. The first case is that $x1$ is a prefix of w. Then $w = x1z$ for some z. In this case, since x has an even number of both 0s and 1s, so must z. Hence w is in $L(R1R)$ and therefore in $L(R(1 \cup 01(11) * 0)R)$.

2. The second case is that $x0$ is a prefix of w. Now, $x00$ cannot be a prefix of w, since $x00$ has an even number of both 0s and 1s, but we know that x is the longest prefix of w with an even number of both 0s and 1s. Therefore, $x01$ must be a prefix of w. The string $x01$ has an odd number of 0s and an odd number of 1s; hence another 0 must occur in w after the prefix $x01$. Therefore, $w = xyz$ where y is of the form $011 * 0$. But in fact, y must contain an odd number of 1s: if this were not true then xy would be a prefix of w containing an even number of 0s and 1s and would be longer than x, but we know that x is the longest prefix of w containing an even number of 1s. Thus, y is a string with a 0 at each end and an odd number of 1s between these two 0s, which means that y must be in $L(01(11) * 0)$. Therefore, y is also in $L(1 \cup 01(11) * 0)$. We have now shown that $w = xyz$ where x has an even number of both 0s and 1s and y is in $L(01(11) * 0)$. Since the string xy has an even number of 0s and an odd number of 1s, it must be that z has both an even number of 0s and an even number of 1s. Thus, $w = xyz$ where x is in $L(R)$, y is in $L(1 \cup 01(11) * 1)$ and z is in $L(R)$. Therefore, $w = xyz$ is in $L(R(1 \cup 01(11) * 1)R)$ and we are done.

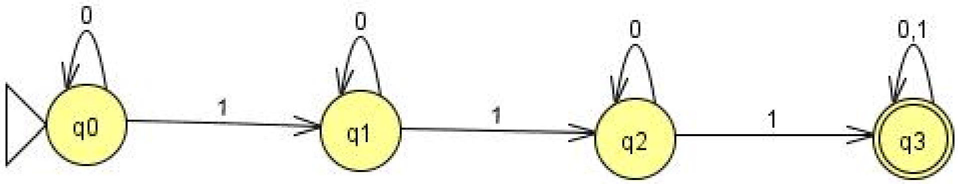

Figure 1: DFA for problem 2.b.

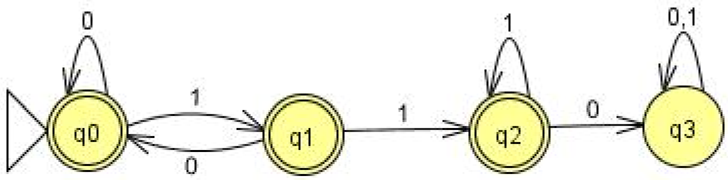

Figure 2: DFA for this problem 2.f.

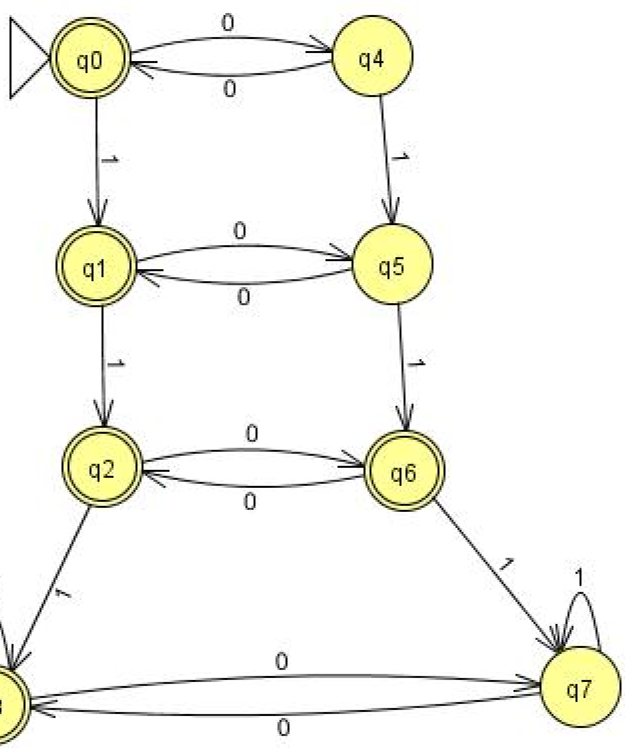

Figure 3: DFA for problem 2.l.