CPSC 536F March 29

- Normal Extension Theorem
- Cayley Graphs
- Galois Correspondence ← A bit theoretical.

Back to eigenvalues } for as much time as we have

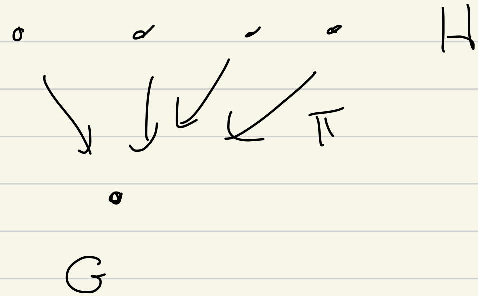
Normal Extension Theorem

(It's analog of the classical theorem in Galois theory for graphs.)

Theorem: Let $\pi : H \to G$ be a covering map of connected graphs. Then there is a covering map $\pi' : H' \to H$ s.t. $\pi \circ \pi'$ is Galois.

$$\begin{tikzcd}
H' \arrow[d, "\pi'"] \arrow[dd, bend left=60, "\pi \circ \pi'"] \\
H \arrow[d, "\pi"] \\
G
\end{tikzcd}$$

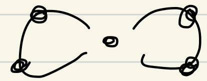
"Galois" $H \xrightarrow{\pi} G$ when H is not connected (but G is) is defined: for any $v, v' \in V_H$ s.t. $\pi(v) = \pi(v')$, there is a map is $\mathrm{Aut}_H(G)$ taking v to v'.

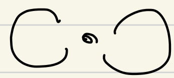


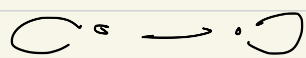
all isolated vertices. Then

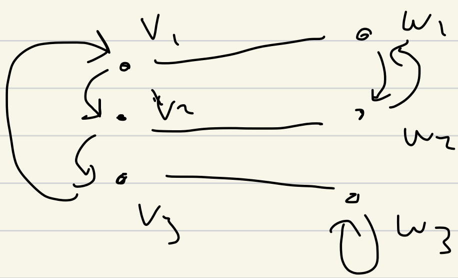
$\mathrm{Aut}_G(H)$ = all permutations of vertices, size $4!$, not 4.

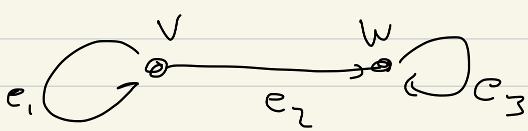
Example:

simple graph 

graph, G = 

related 

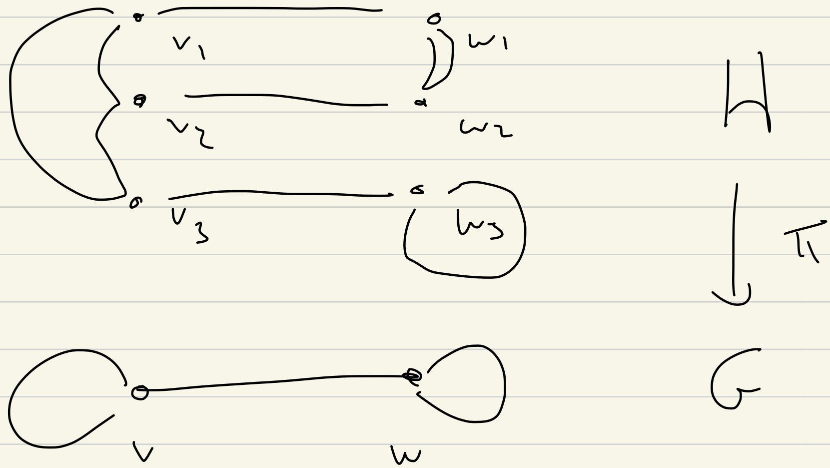




G

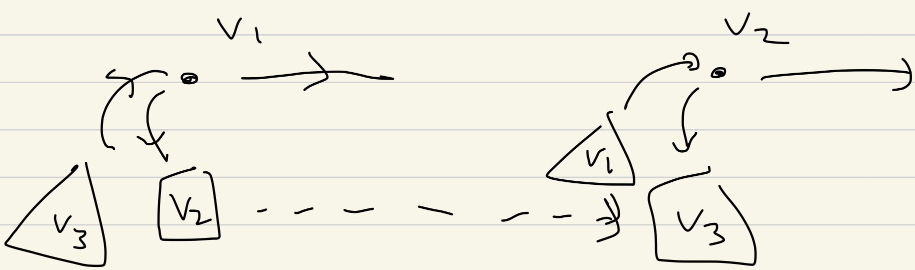
with same directed edges

All directed edges



Notice: there is no automorphism

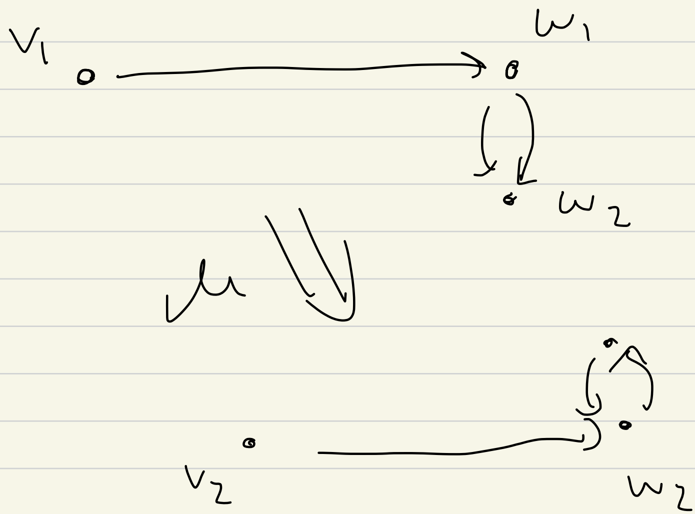
$\mu \in Aut_G^H(H)$ s.t. $\mu(v_1) = v_2$



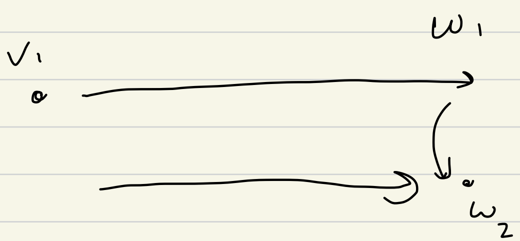
$\mu(v_1) = v_2 \Rightarrow \mu(v_2) = v_3$

and $\mu(v_3) = v_1$

But $\mu(v_1) = v_2$



$\Rightarrow \mu(w_1) = w_2, \quad \mu(w_2) = w_1$



gives same local/global

$$\begin{tikzcd} v_1 \arrow[r] & w_1 \arrow[d] \\ v_2 \arrow[r] & w_2 \end{tikzcd}$$

So $\mu(v_1) = v_2 \Rightarrow \mu(v_2) = v_1$

Nicer (Philip): Looking at

$\mathrm{Aut}_G(H)$ must contain

$\tau(w_1) = w_2$

$\tau(w_2) = w_1$

$\tau(w_3) = w_3$

Recall a little group theory

<u>Def</u> $\mathcal{G}$ is a <u>group</u> if

(1) $\mathcal{G}$ is a <u>set</u>

(2) $\mathcal{G}$ is endowed with multiplication

$$(g_1, g_2) \longmapsto g_1 g_2$$

s.t. (a) associative

$$(g_1 g_2) g_3 = g_1 (g_2 g_3)$$

(b) there is an <u>inverse</u> <u>map</u>!

$\exists$ identity $e \in \mathcal{G}$ s.t.

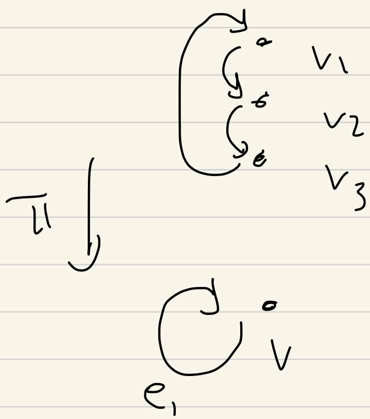
$$eg = ge = g \quad \forall g \in \mathcal{G}$$

$\exists$ map $g \mapsto g^{-1}$ s.t. $gg^{-1} = g^{-1}g = e$

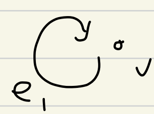
(1) $\mathrm{Aut}_G(H) \hookrightarrow$ Permutations on V_H

finite group

(2) $|\mathrm{Aut}_G(H)| = 3$ then it can't contain a subgroup of order 2.



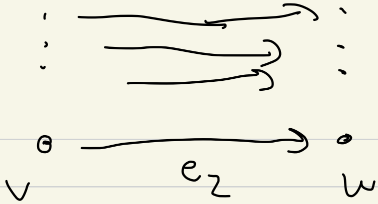
this is how directed edges map to e_1

To 

we associate

map
$$v_1 \mapsto v_2$$
$$v_2 \mapsto v_3$$
$$v_3 \mapsto v_1$$

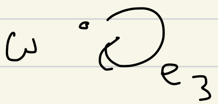
to



to e_2 we associate

$v_1 \mapsto w_1$
$v_2 \mapsto w_2$
$v_3 \mapsto w_3$

to e_3



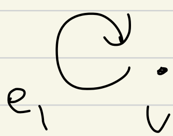
we associate

$w_1 \mapsto w_2$
$w_2 \mapsto w_1$
$w_3 \mapsto w_3$

rule

(v_i, v_j, v_k) to

$$\left(\mathrm{perm}_{e_1}(v_i), \mathrm{perm}_{e_1}(v_j), \mathrm{perm}_{e_1}(v_k)\right)$$



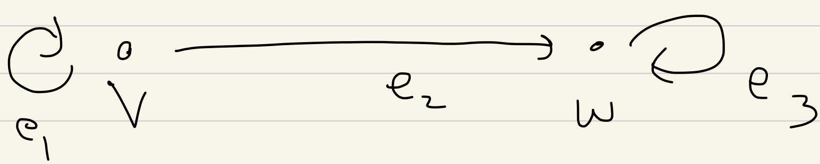
Covering map: if $\pi : H \to G$ is d-to-1,
want $d!$-to-1

$$(v_1, v_2, v_3) \longrightarrow (w_1, w_2, w_3)$$
$$(v_1, v_3, v_2) \longrightarrow (w_1, w_3, w_2)$$
$$(v_2, v_1, v_3) \longrightarrow (w_2, w_1, w_3)$$
$$(v_2, v_3, v_1) \longrightarrow (w_2, w_3, w_1)$$
$$(v_3, v_1, v_2) \longrightarrow (w_3, w_1, w_2)$$
$$(v_3, v_2, v_1) \longrightarrow (w_3, w_2, w_1)$$

H'

$\downarrow$

G



Full graph directed edges to edges

$$\begin{array}{ccc}
\boxed{(v_1, v_2, v_3)} & \text{———} & (w_1, w_2, w_3) \\
(v_1, v_3, v_2) & \text{———} & (w_1, w_3, w_2) \\
(v_2, v_1, v_3) & \text{———} & (w_2, w_1, w_3) \\
(v_2, v_3, v_1) & \text{———} & (w_2, w_3, w_1) \\
\boxed{(v_3, v_1, v_2)} & \text{———} & (w_3, w_1, w_2) \\
(v_3, v_2, v_1) & \text{———} & (w_3, w_2, w_1)
\end{array}$$

μ

=

Claim: H' is Galois:

e.g. $\mu\big((v_1, v_2, v_3) \big) = (v_3, v_1, v_2)$

can we extend μ to all of H?

$\mu(v_1, v_2, v_3) = (v_3, v_1, v_2)$

forces

$\mu(v_2, v_3, v_1) = (v_1, v_2, v_3)$

forces

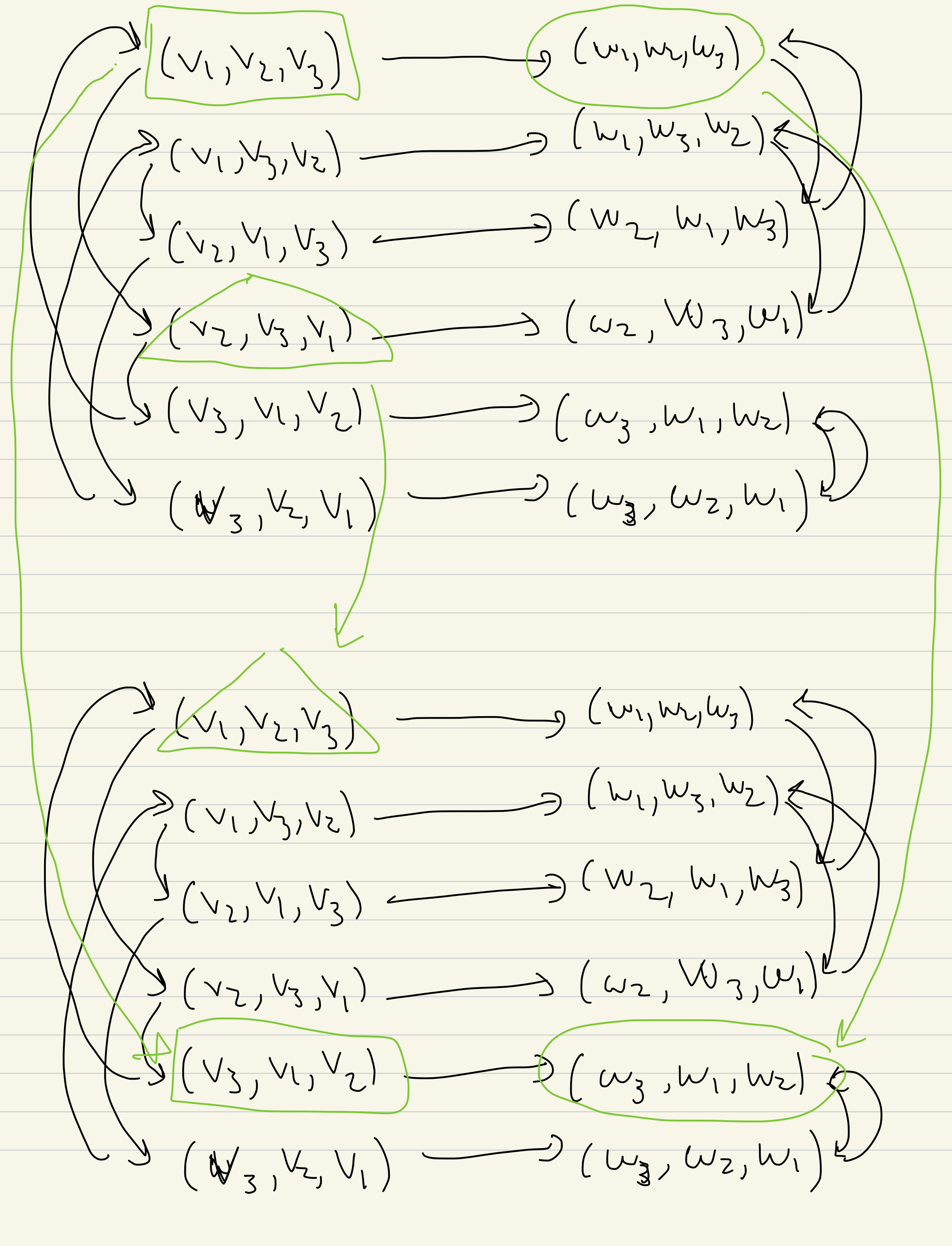
$\mu(w_1, w_2, w_3) = (w_3, w_1, w_2)$

etc.

What is the general rule?

(Break)

Say that $i, j, k \in \{1, 2, 3\}$ distinct and that for $i = 1$, $j = 2$, $k = 3$



(v1, v2, v3)
(v1, v3, v2)
(v2, v1, v3)
(v2, v3, v1)
(v3, v1, v2)
(v3, v2, v1)
(w1, w2, w3)
(w1, w3, w2)
(w2, w1, w3)
(w2, w3, w1)
(w3, w1, w2)
(w3, w2, w1)
(v1, v2, v3)
(v1, v3, v2)
(v2, v1, v3)
(v2, v3, v1)
(v3, v1, v2)
(v3, v2, v1)
(w1, w2, w3)
(w1, w3, w2)
(w2, w1, w3)
(w2, w3, w1)
(w3, w1, w2)
(w3, w2, w1)

We want for $i=1$, $j=2$, $k=3$

$$\mu(v_i, v_j, v_k)$$

$$= (v_k, v_i, v_j)$$

But

edge are e_1

$$(v_1, v_2, v_3) \longrightarrow (v_2, v_3, v_1)$$

obtained since

$$v_1 \longmapsto v_2 = \rho_{e_1}(v_1)$$

$$v_2 \longmapsto v_3 = \rho_{e_1}(v_2)$$

$$v_3 \longmapsto v_1 = \rho_{e_1}(v_3)$$

in edge over e_1

take 1st comp to 2nd
" 2nd comp to 3rd
" 3rd " to 1st

Claim:

$(v_i, v_j, v_k) \longmapsto (v_k, v_i, v_j)$

works (is forced) for all i, j, k

under e_1

$\left(\mathrm{perm}_{e_1}(v_i), \mathrm{perm}(v_j), \mathrm{perm}(v_k) \right)$

So Symm_3 = permutations on 3 elements

acts on (x, y, z)

$(x, y, z) \longrightarrow (z, x, y)$

Claim:

Symm_3 acts on components:

$$(V_i, V_j, V_k) \xrightarrow{\text{transpose } 2^{nd} \,\&\, 3^{rd}} (V_i, V_k, V_j)$$

extends to a global automorphism of $H' \longrightarrow G$

Hence there exist a map

$$(V_1, V_2, V_3) \longrightarrow \text{any other ordering } (V_i, V_j, V_k), \; i, j, k \text{ distinct}$$

2^{nd} claim: there is a map

$$\begin{tikzcd} H' \arrow[d] \\ H \end{tikzcd} \quad \text{s.t.} \quad \begin{tikzcd} H' \arrow[d] \arrow[dd, bend left=60, "\text{is this map}"] \\ H \arrow[d] \\ G \end{tikzcd}$$

namely

$$H' \xrightarrow{\text{proj onto } 1^{st} \text{ comp.}} H$$

$$(v_i, v_j, v_k) \longmapsto v_i$$

Last part: if $H' \longrightarrow G$ is Galois: there is a 1-1 correspondence

$$(v_1, v_2, v_3) \longrightarrow (w_1, w_2, w_3)$$

$$(v_1, v_3, v_2) \longrightarrow (w_1, w_3, w_2)$$

$$(v_2, v_1, v_3) \longrightarrow (w_2, w_1, w_3)$$

$$(v_2, v_3, v_1) \longrightarrow (w_2, w_3, w_1)$$

$$(v_3, v_1, v_2) \longrightarrow (w_3, w_1, w_2)$$

$$(v_3, v_2, v_1) \longrightarrow (w_3, w_2, w_1)$$

$\downarrow \quad \downarrow \quad \downarrow$ covering map?

$$v_1 \longrightarrow w_1$$

$$v_2 \longrightarrow w_2$$

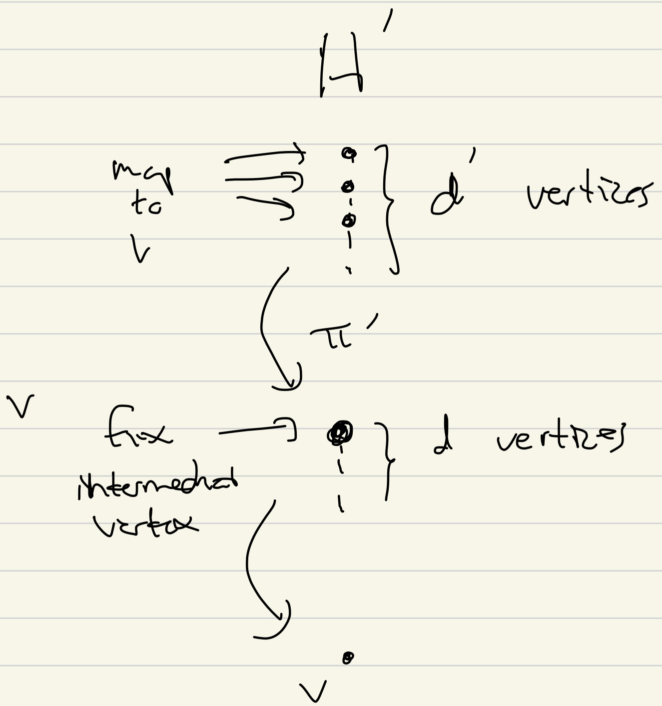
$$v_3 \longrightarrow w_3$$

$$\dot{v} \longrightarrow \dot{w}$$

Last part: if $H' \longrightarrow G$ is Galois: there is a 1-1 correspondence between

$$H' \longrightarrow H \longrightarrow G$$

and subgroups of $\mathrm{Aut}_G(H)$



Look at $\mu \in \mathrm{Aut}_G H$ s.t. $\mu(\pi'^{-1}(v))$ to itself

Last part:

$$H \longrightarrow G \quad \text{étale}$$

then
$$\lambda_1(H) \le \lambda_1(G)$$

uses lifting lemma.

Given λ_2

Wilf

Hoffman

to chromatic(G)