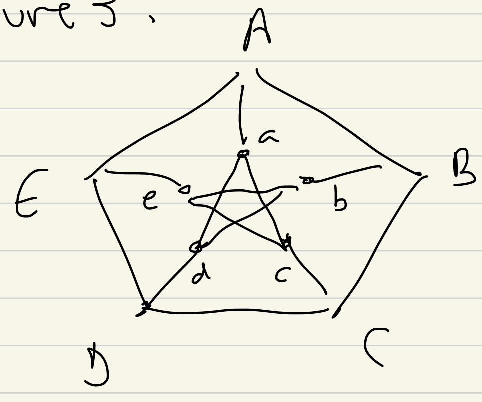
March 24, 2022 CPSC 536F

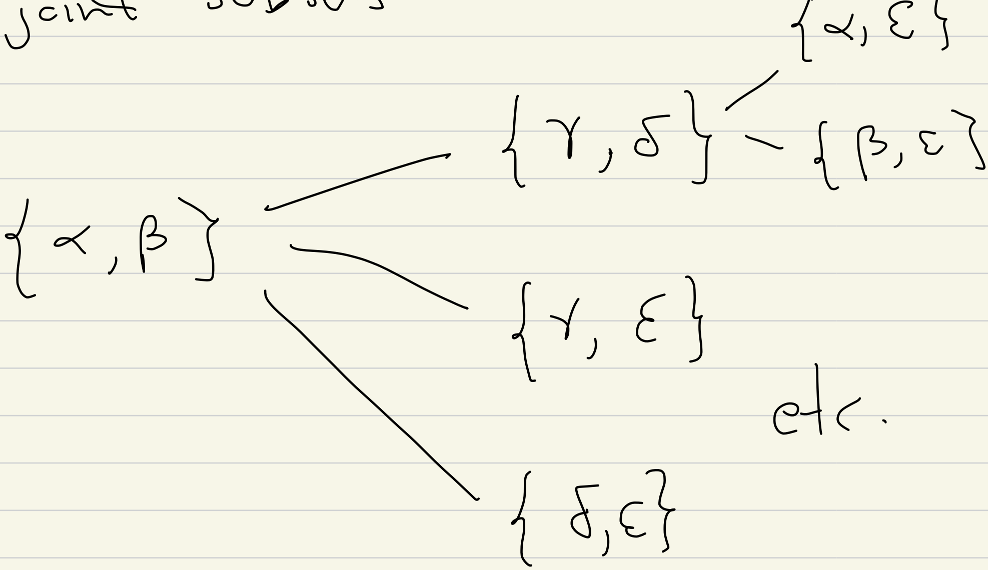
A common conceptual small examples:

- 2 state Markov chain

- Kempe-Petersen graph: important because it's often a counterexample to reasonable conjectures:

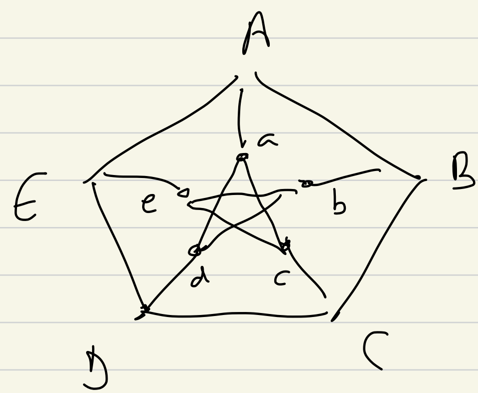


Also: If take all 2 element subsets of set of size 5, $\{\alpha, \beta, \gamma, \delta, \varepsilon\}$ as V_G, edges connect disjoint subsets

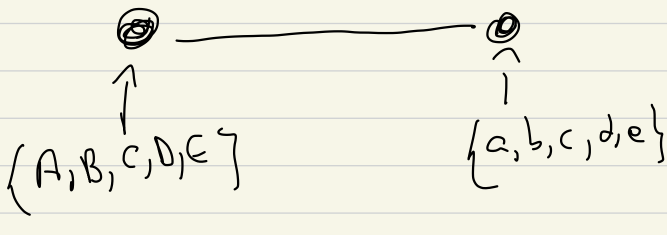


this is a Kneser graph, this is isomorphic to Petersen graph.

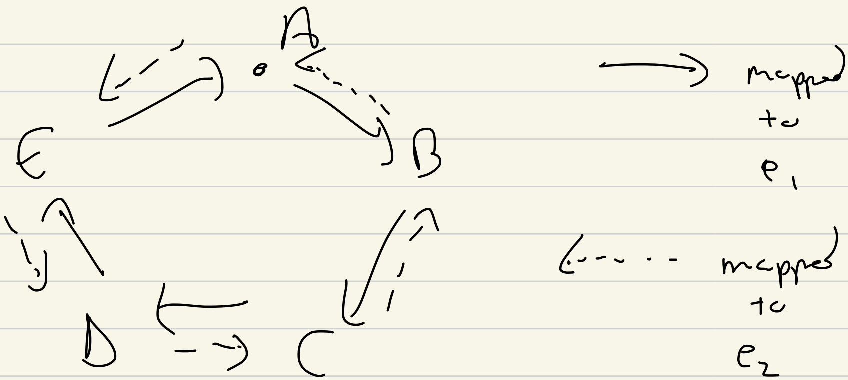
Exercise: Let's compute the eigenvalues/vectors of Petersen graph:



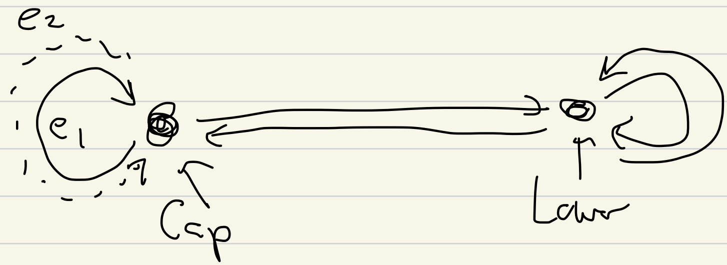
$\pi \big\downarrow$ covering, 5-to-1



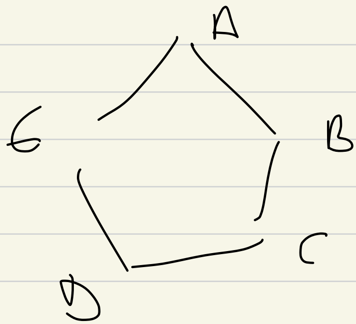
Let's be careful:



$$e_1^{-1} \text{ or } 2e_1 \text{ is } e_2$$



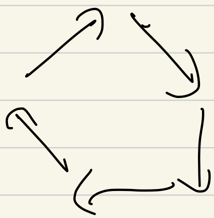
$$\begin{array}{c c} & \begin{array}{cc} \text{Cap} & \text{Lower} \end{array} \\ \begin{array}{c} \text{Cap} \\ \text{Lower} \end{array} & \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{array}$$

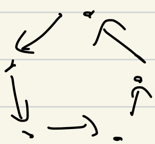


given by

p_1 : permutation:

$A \to B$
$B \to C$
$C \to D$
$D \to E$
$E \to A$

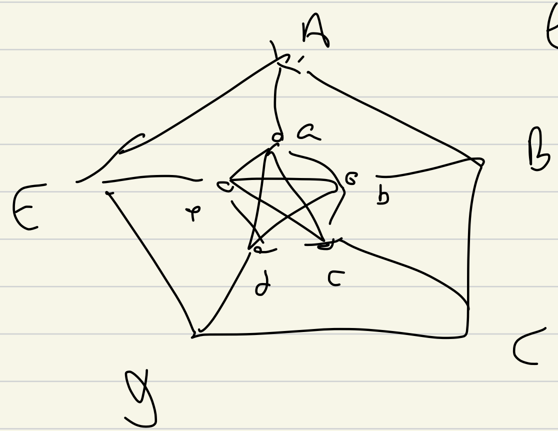




$$\left.\begin{array}{l} p_1 \\ \\ p_1^{-1} \end{array}\right\} \text{gives}$$

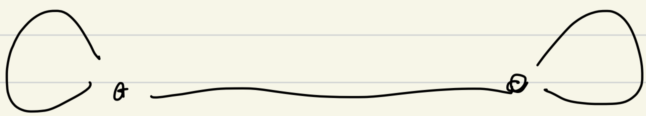
Claim: The map

3-regular



$$H \xrightarrow{\pi} G$$

3-regular



is Galois, with symmetry group $C_5 = \mathbb{Z}/5\mathbb{Z}$ = integers mod 5

$$\begin{tikzcd} H \arrow[rr, "\mu"] \arrow[dr, "\pi"'] & & H \arrow[dl, "\pi"] \\ & G & \end{tikzcd}$$

$\mu(A) = B$, $\mu(a) = b$

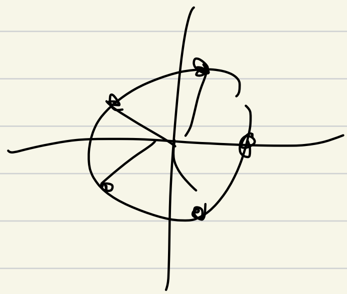
$\mu(B) = C$ $\mu(b) = c$

$\mu(C) = D$ c d

$\mu(D) = E$ d e

$\mu(E) = A$ e a

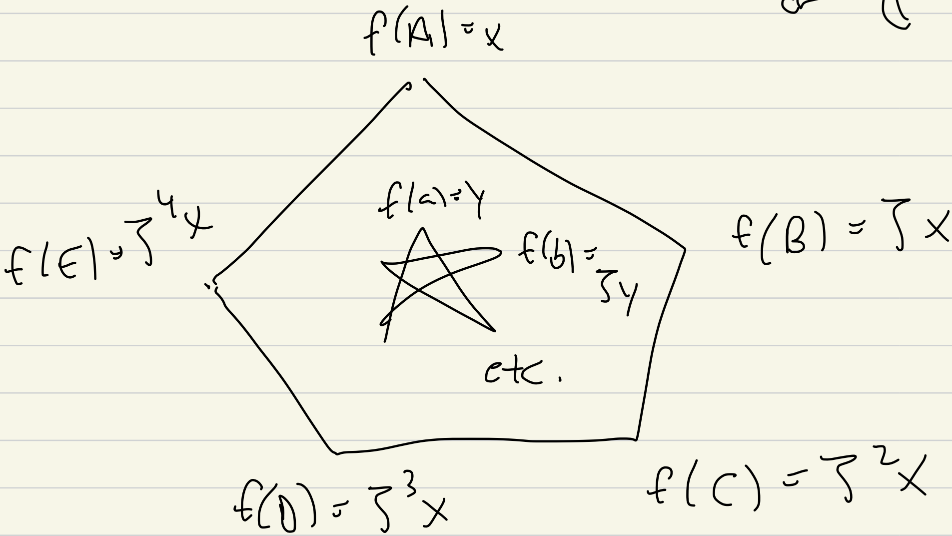
Claim: Let $\zeta^5 = 1$



$\mathbb{C}$

$\zeta = 1$

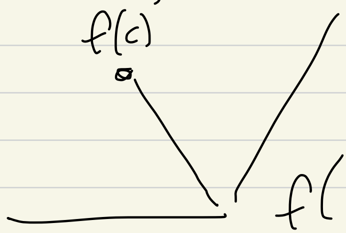
Look at functions $x, y \in \mathbb{R}$ or $\mathbb{C}$



Look at

V.S.

$$\left(A_{\text{Petersen}}\, f\right)(C) = \bigcirc\, f(C)$$



$f(c) = \zeta^2 x$

$f(B) = \zeta \cancel{y} = \zeta^{-1} f(C)$

$f(C) = \zeta^2 \cancel{x}$

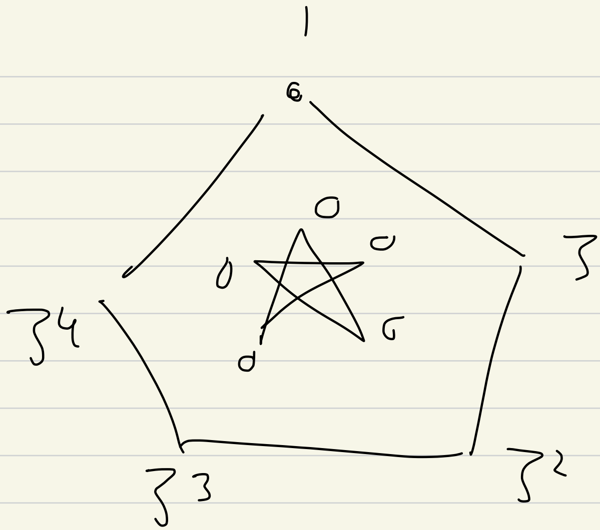
$f(D) = \zeta^3 \cancel{x}$

$= \zeta\, f(C)$

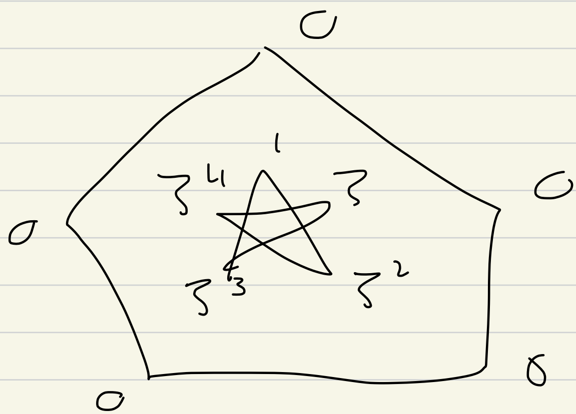
$$f(D) + f(B) + f(C)$$

$$= \zeta\, f(C') + \zeta^{-1} f(C') + \text{ratio}\ \frac{f(c)}{f(C)}\, f(C)$$

vect_1^{ζ}



vect_2^{ζ}



$$(A_{\text{Petersen}})(\text{vect}_1) = (\zeta+\zeta^{-1})(\text{vect}_1) + \text{vect}_2$$

$$'' \qquad \text{vect}_2 = \text{vect}_1 + (\zeta^2+\zeta^{-2})\,\text{vect}_2$$

So by using symmetry, μ, of order 5 plus "guesswork", for each $\zeta^5 = 1$ we get

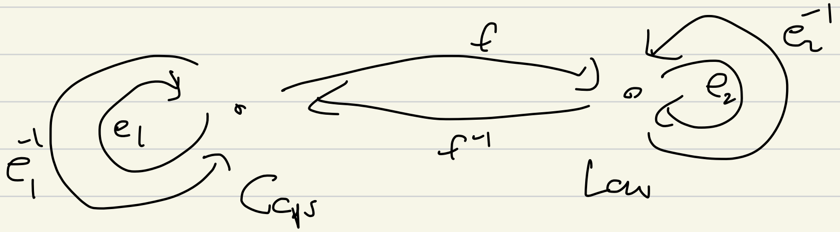
$$\left[A_{\text{Petersen}} \right] \begin{bmatrix} \text{vect}_1^{\zeta} \\ \text{vect}_2^{\zeta} \end{bmatrix} = \begin{bmatrix} \zeta + \zeta^{-1} & 1 \\ 1 & \zeta^2 + \zeta^{-2} \end{bmatrix} \begin{bmatrix} \text{vect}_1^{\zeta} \\ \text{vect}_2^{\zeta} \end{bmatrix}$$

We claim: the eigenvectors of

$$\begin{bmatrix} \zeta + \zeta^{-1} & 1 \\ 1 & \zeta^2 + \zeta^{-2} \end{bmatrix}$$

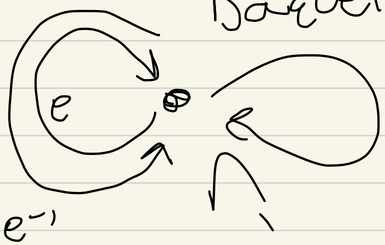
ranging over ζ give ON set of eigenvectors of A_{Petersen}

Remark!



has 2-1 covering map

Bouquet



$f = f^{-1} \therefore 1f$

$e \neq e^{-1} \therefore 2e$

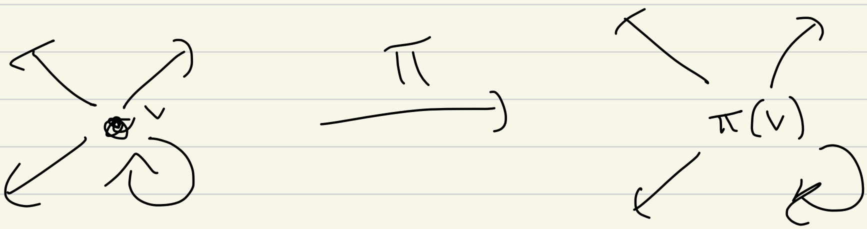
3-regular

$$A_{\text{Bouquet}} = [3]$$

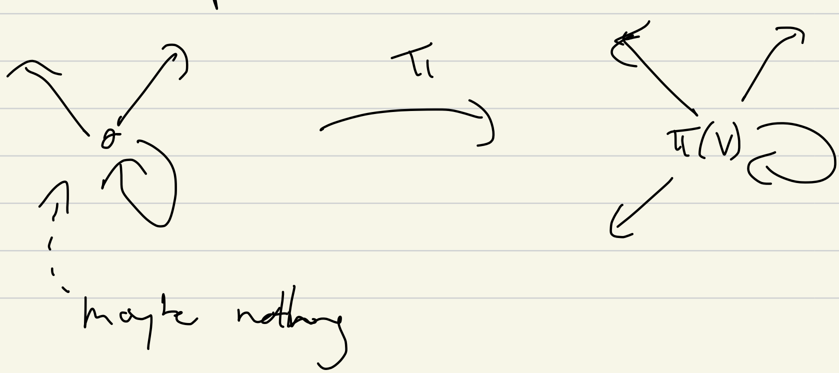
$$A_{\text{any } d\text{-regular graph on one vertex}} = [d]$$

(Exercise) The composition of 2 covering maps is covering, and of 2 étale maps " étale

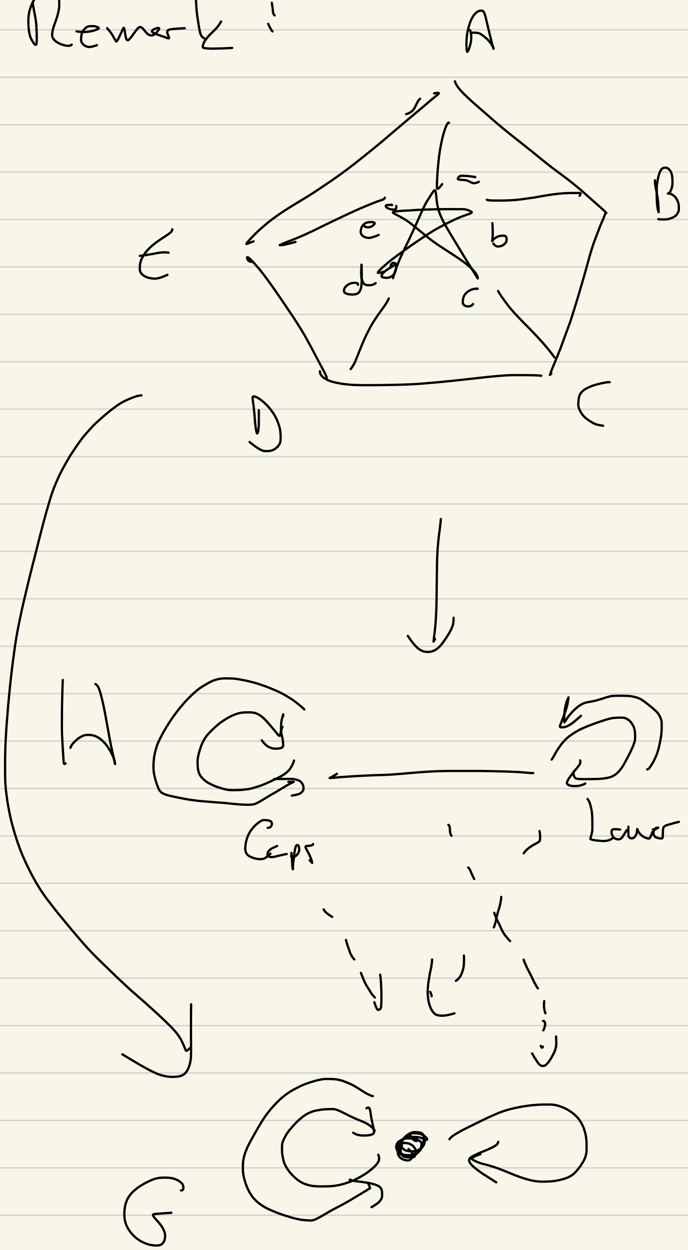
covering map



étale map



Remark:



Petersen

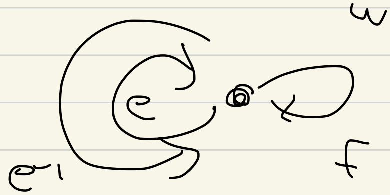
$|\mathrm{Aut}_H(\mathrm{Petersen})| = 5$

$|\mathrm{Aut}_G H| = 2$

covering

But $\mathrm{Aut}_G(\mathrm{Petersen}) = 5$

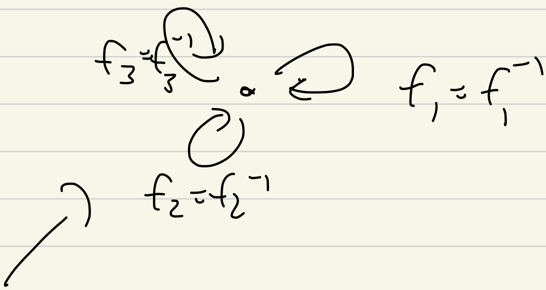
Exercise: If G is 3-regular, but not



with 1 vertex

$f = f^{-1}$,

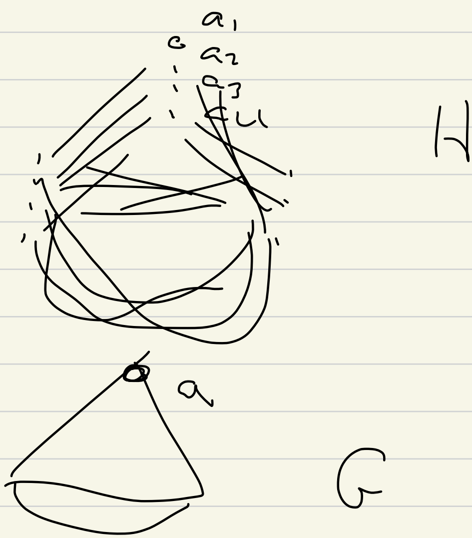
OR



G' has no covering map

Petersen $\longrightarrow$ G'

Fact: If H, G connected (di)graphs and $\pi : H \to G$ covering,



$$|\mathrm{Aut}_G H| \leq d$$

where π is d-to-1.

The "lifting lemma": Let

$\pi : H \to G$ be a covering

map of graphs (for simplicity).

$$\begin{array}{c} H \\ \pi \big\downarrow \\ G \end{array}$$

Consider a walk in G:

$$w = (v_0, e_1, v_1, e_2, \ldots, e_k, v_k)$$

$$\underset{v_0}{\circ} \xrightarrow{e_1} \underset{v_1}{\circ} \xrightarrow{e_2} \underset{v_2}{\circ} \quad \cdots \quad \xrightarrow{e_k} \circ \, v_k$$

(so all $v_i \in V_G$, $e_i \in E^{dir}_G$,
$te_i = v_{i-1}$, $he_i = v_i$)

Take $v' \in V_H$ s.t. $\pi(v') = v_0$

$$\begin{tikzcd}
H \arrow[d, "\pi"'] & v' \arrow[d, "\pi"'] & & & \\
\text{in } G! & v_0 \arrow[r, "e_1"'] & v_1 \arrow[r, "e_2"] & \cdots \arrow[r, "e_k"] & v_k
\end{tikzcd}$$

Then there is a unique walk in
H

$$v' = v'_0 \xrightarrow{e'_1} v'_1 \xrightarrow{e'_2} \cdots \xrightarrow{e'_k} v'_k$$

that maps to

$$v_0 \xrightarrow{e_1} \cdots \xrightarrow{e_k} v_k$$

(i.e. $v_i = \pi(v'_i)$, $e_i = \pi(e'_i)$)

Proof: Induction on k:

$$\begin{tikzcd} v_0' = v' \arrow[r, "e_1'"] \arrow[d, "\pi"'] & v_1' \arrow[d, dashed, "\pi"] \\ v \arrow[r, "e_1"'] & v_1 \end{tikzcd}$$

π maps $\{\text{edges with tail in } v_0'\} \xrightarrow{\text{bijectively}} \{\text{'' '' '' '' } v_0\}$

Cor: If $\pi : H \xrightarrow{\text{covering}} G$,

H (and G) are connected, ~~then~~

$v', v'' \in V_H$ s.t. $\pi(v') = \pi(v'')$

$$\begin{tikzcd} v' \arrow[d, bend right, "\mu"'] \\ v'' \arrow[d] \\ v \end{tikzcd}$$

$v \in V_G$ $\quad v = \pi(v') = \pi(v'')$

then there is at most one map

$\mu : H \to H$ that is an isomorphism

and lies in $\mathrm{Aut}_G H$, i.e. μ

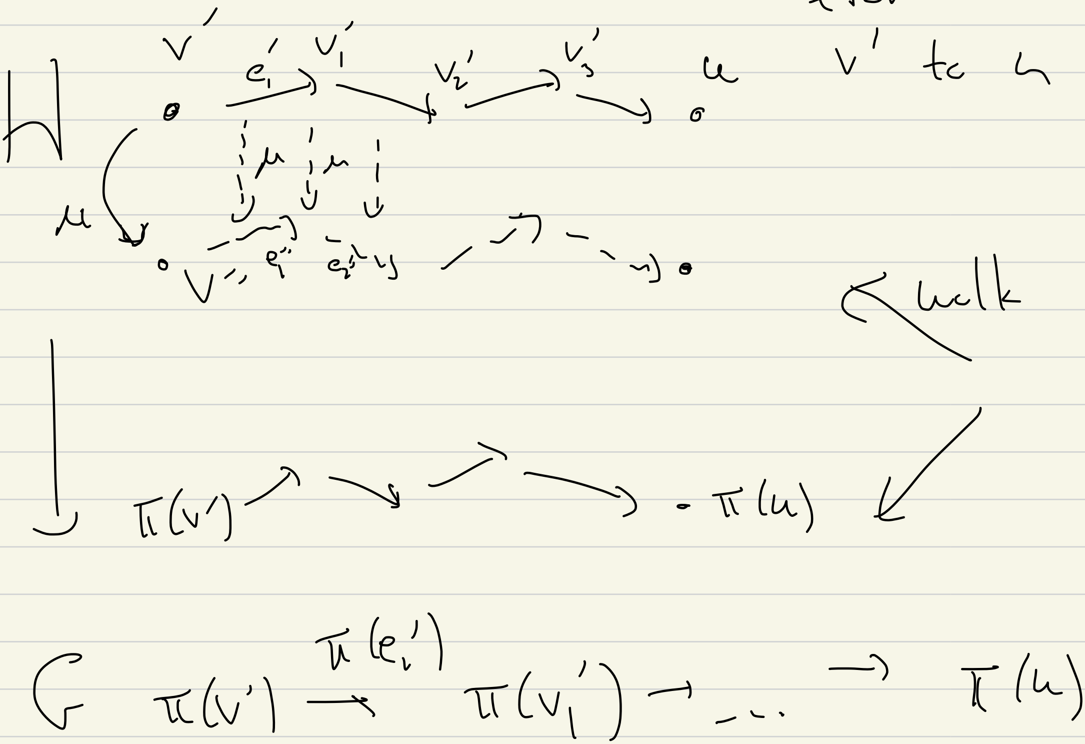
respects the covering map, i.e.

$$\begin{tikzcd} H \arrow[rr, "\mu"] \arrow[dr, "\pi"'] & & H \arrow[dl, "\pi"] \\ & G & \end{tikzcd}$$

, i.e $\pi \mu = \pi$

so μ is local bijection

Proof! If $u \in V_H$, take a walk from v' to u

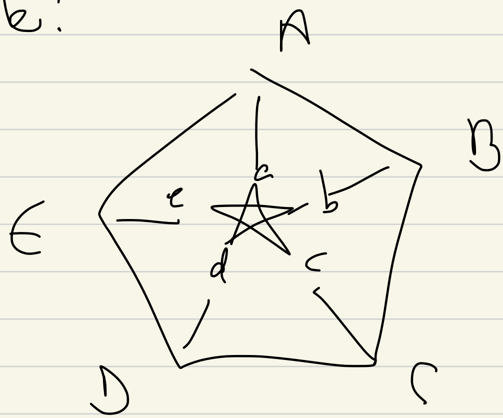


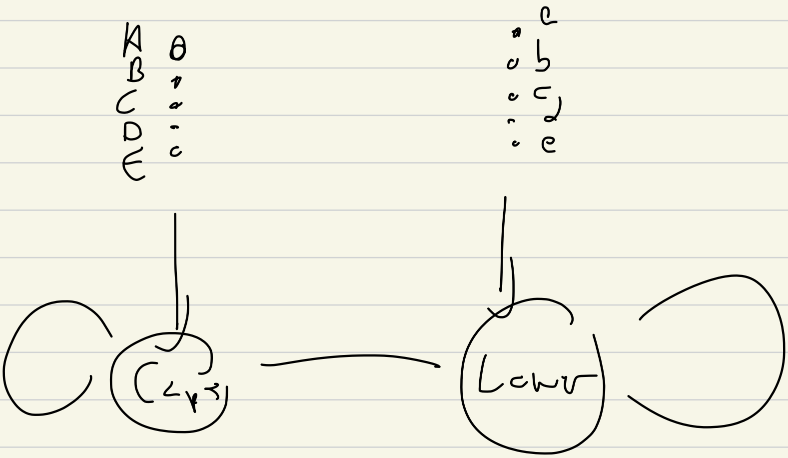
the lifting lemma gives walk over walk in G starting at v''

By induction μ must map

v' to v'', e_1' to e_1'', ...

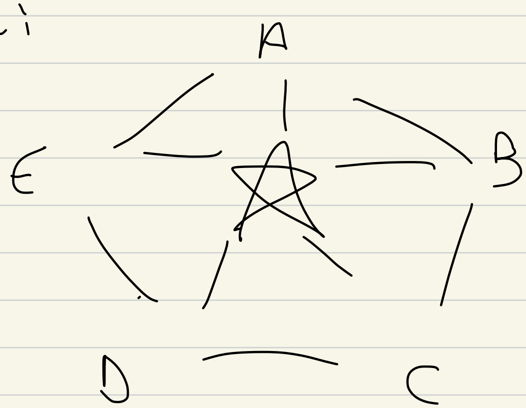
Example:

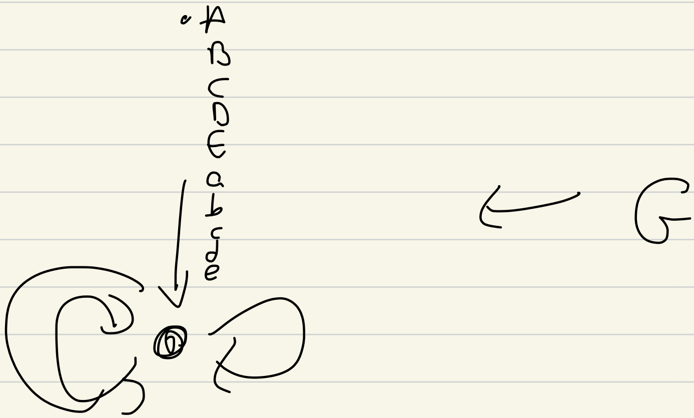




If $\mu: A \to C$ forces $a \mapsto c$, $B \mapsto D$, $E \mapsto B$ etc.

Claim:





Exercise! Verify there is no $\mu \in \mathrm{Aut}_{G}(\text{Petersen})$ taking

A to any small letter vertex.