CPSC 536F March 22, 2022

Last time:

- Defined <u>covering</u> maps and <u>étale</u> maps.

on hold

- Stated: If H, G are graphs s.t. there exists an étale map $H \to G$, then

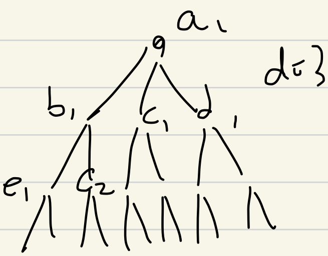
$$\lambda_1(H) \leq \lambda_1(G)$$

with equality if $H \to G$ is a covering map

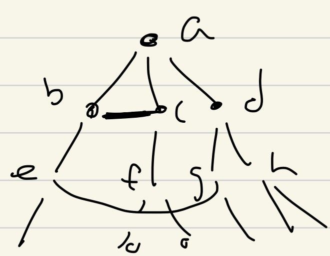
- Question: What are examples of étale and covering maps?

For Alon-Boppana thm:

Truncated at level ℓ d-reg tree $\longrightarrow$ Vertices distance $\le \ell$ of d-regular graph



$\longrightarrow$



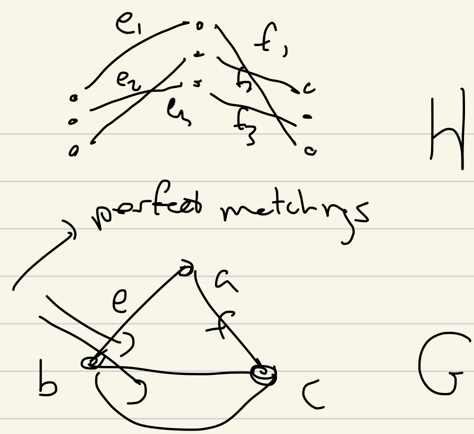
$$H \xrightarrow{\text{étale}} G$$

$\Rightarrow$

$$\lambda_1(H) \le \lambda_1(G)$$

uses "lifting lemma"

Examples!



3 to 1 covering map

(1) put 3 "copies" of each vertex in G "above"

(2) for each edge in G, put a perfect matching "above".

Galois theory of graphs!

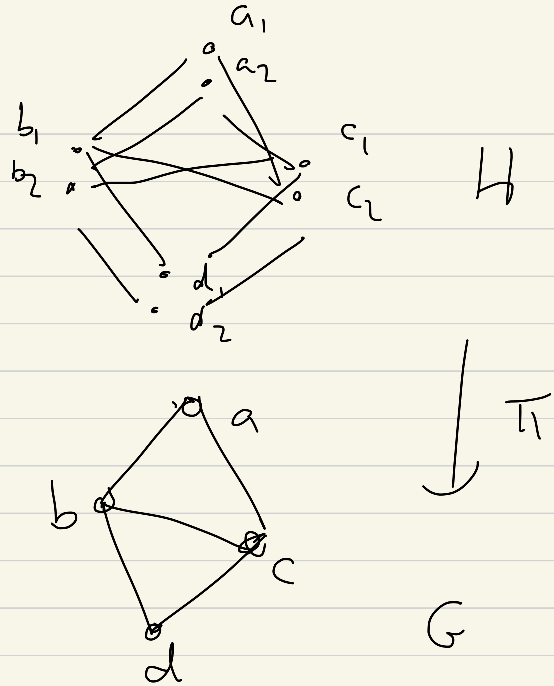
If H is any graph,

$$\mathrm{Aut}(H) = \left\{ \cancel{\phi}: H \to H \text{ that are isomorphisms} \right\}$$

If $\pi : H \to G$ is any morphism,

$$\mathrm{Aut}_G(H) = \left\{ \mu \in \mathrm{Aut}(H) \text{ s.t. } \begin{tikzcd} H \arrow[rr, "\mu"] \arrow[dr, "\pi"'] & & H \arrow[dl, "\pi"] \\ & G & \end{tikzcd}, \quad \text{i.e. } \pi \circ \mu = \pi \right\}$$

Example: Say $\pi : H \to G$ is a 2-to-1 covering map:

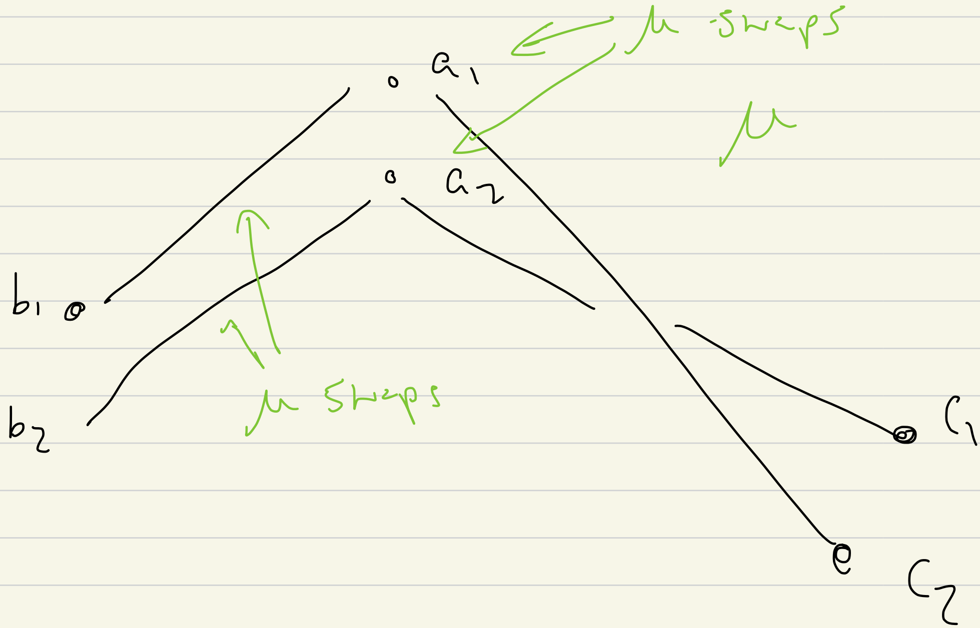


If $\mu : V_H \to V_H$

$E_H^{dir} \to E_H^{dir}$

μ = swap with other thing mapping to same vertex / edge of G

Then $\mu \neq id$, but $\pi \mu = \pi$



Thm: If G, H connected, $\pi : H \to G$ is a k-to-1 covering map, $|Aut_G(H)| \leq k$

<u>Def</u>: In this theorem, if

$$|\mathrm{Aut}_G(H)| = k$$

we say $\pi : H \to G$ is

Galois.

Remark above: If $\pi : H \to G$

is 2-to-1, then π is Galois.

Remark: If $\pi : H \to G$ is of

degree 2, then

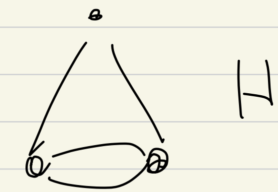
eigenpairs of A_H (Adjacency of H)

can be obtained from A_G, $\widehat{A}_G$

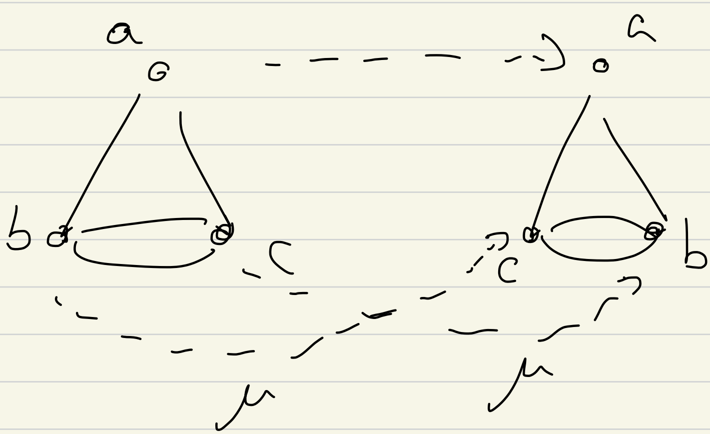
Give, more generally a graph H, with symmetry of order 2:

Homework:

Warm up to Galois theory



H has symmetry of order 2



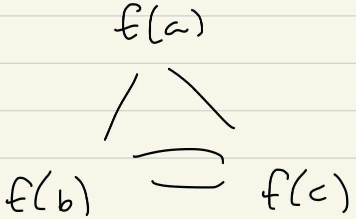
$$A_H = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \end{array}$$

what are λ_j of A_H ?

Say $f : V_H \to \mathbb{R}$

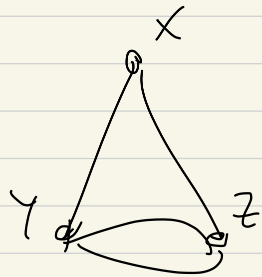
is (1) even if $f_\mu = f$

(2) odd " $f_\mu = -f$



: $f = \dfrac{f + f_\mu}{2} + \dfrac{f - f_\mu}{2}$

$$f: \begin{array}{l} f(a) = x \\ f(b) = y \\ f(c) = z \end{array}$$



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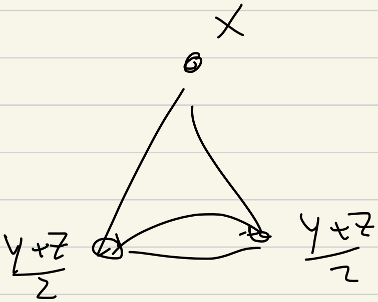

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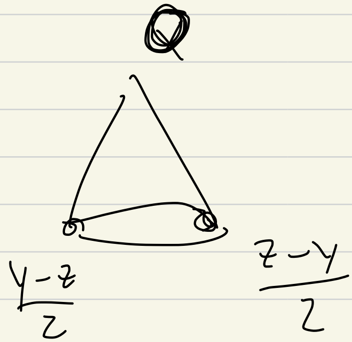
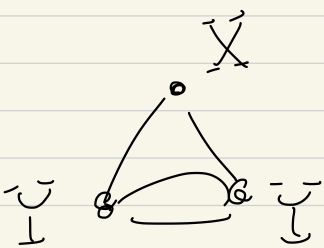
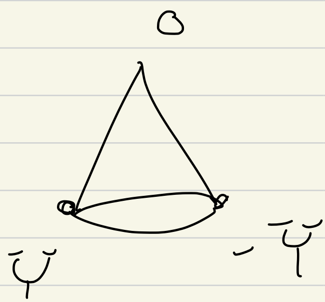


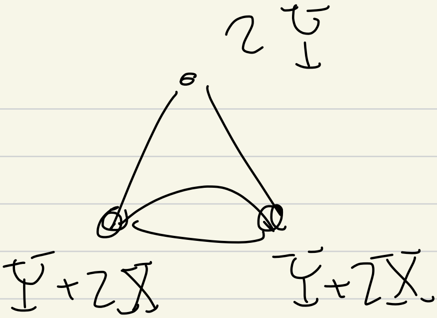
More simply



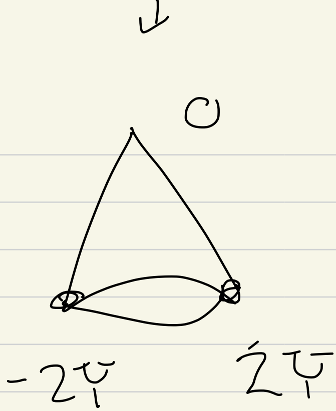
$\downarrow A_H(\sim)$



A_H



"even"



"odd"

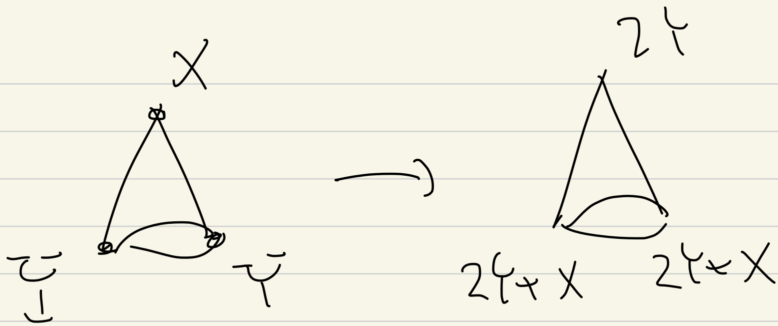
A_H : even $\longrightarrow$ even
odd $\longrightarrow$ odd

as a matrix

$$\begin{array}{c|cc} & \text{even} & \text{odd} \\ \hline \text{even} & \left[\begin{array}{c} \\ \end{array}\right. & \begin{array}{c} 0 \end{array} \\ \text{odd} & 0 & \left.\begin{array}{c} \\ \end{array}\right] \end{array}$$

So

or even



so

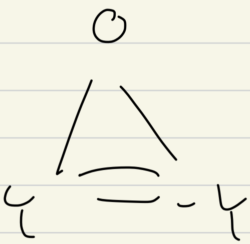
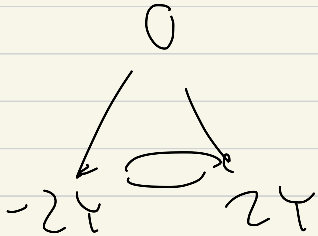
$$A_H \left(\quad \right) = $$

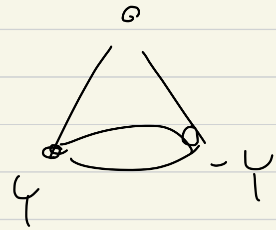
so

$$A_H \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 2Y \\ 2Y + X \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

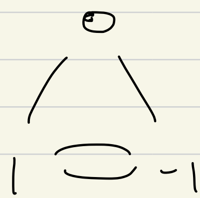
And

$A_H\Bigg($  $\Bigg) =$ 

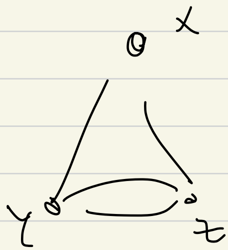
$= (-2)$ 

So

$A_H\Bigg($  $\Bigg)$

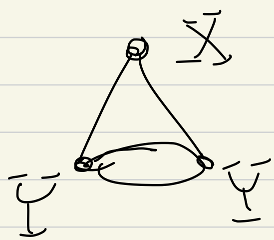
$= (-2)$ 

Think of even functions

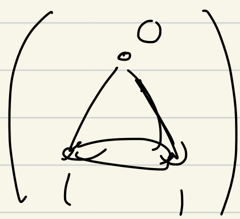


s.t. $\overline{(y \doteq z)}$

Think of any even function

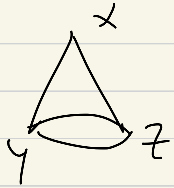


$= X$ 

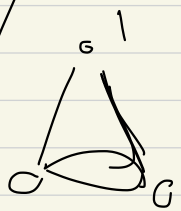
$+ 2^{Y}$ 

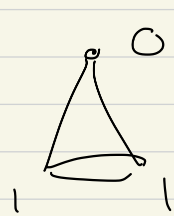
even$_1$ even$_2$

$even_1$, $even_2$ are a basis

for all functions 

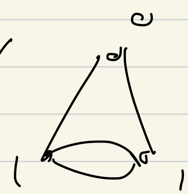
s.t. $y = z$

$A_H\Big($ 

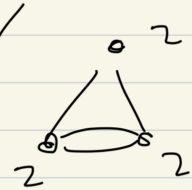
$\Big) = \Big($ 

$\Big)$

$A_H(even_1) = even_2$

$A_H(even_2) = A_H\Big($ 

$\Big)$

$= \Big($ 

$\Big)$

$= 2\, even_1 + 2\, even_2$

So

$A_H \; \text{even}_1 = \qquad \text{even}_2$

$A_H \; \text{even}_2 = 2\,\text{even}_1 + 2\,\text{even}_2$

$A_H(\alpha\, \text{even}_1 + \beta\, \text{even}_2)$

$= (2\beta)\,\text{even}_1 + (2\alpha + 2\beta)\,\text{even}_2$

In basis $\text{even}_1, \text{even}_2$,

$$A_H \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \to \begin{bmatrix} 2\beta \\ 2\alpha + 2\beta \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

HW: Check that this gives

(1) 3 ON eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$

$$A_H \vec{v}_i = \lambda_i \vec{v}$$

(2)
$$(\text{any even}) \circ (\text{any odd}) = 0$$

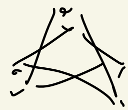
=

<u>Break</u>

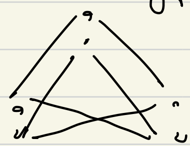
Deg 2 covering map $H \to G$.

Consider 2-to-1 covers

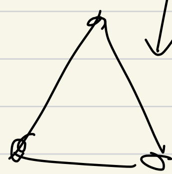
(2-lifts)

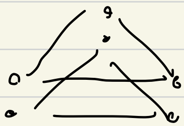


OR



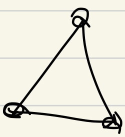
Cycle$_6$





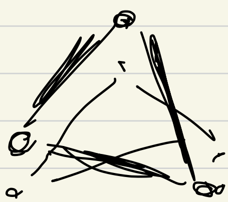
← 2 triangles

Two Cycle$_3$



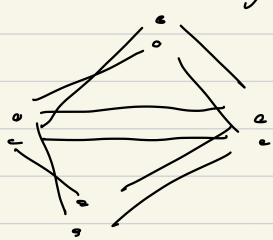
Cycle$_3$

OR



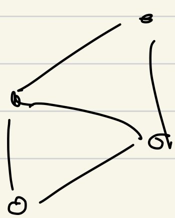
OR

H

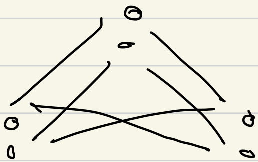


μ reverse fibres

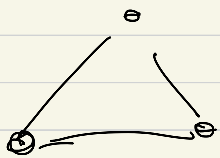
G



Even functions

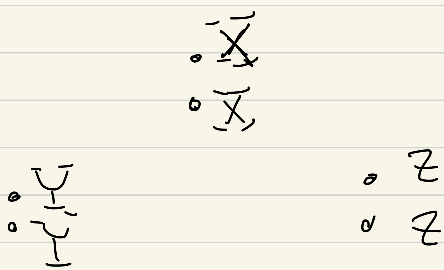


$H \circlearrowleft \mu$

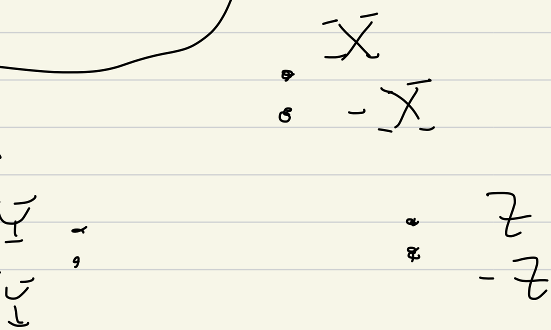


G

Even functions : $V_H \to \mathbb{R}$



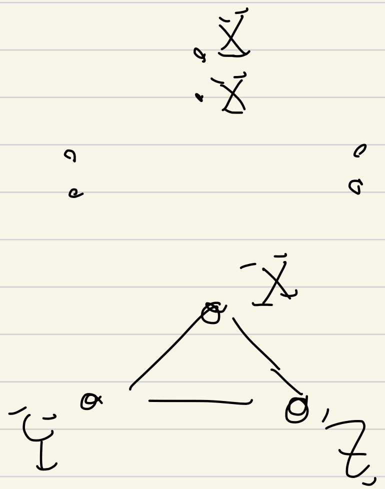
Odd functions



① Even functions, i.e. $f : V_H \to \mathbb{R}$

s.t. $f\mu = f$

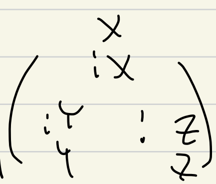
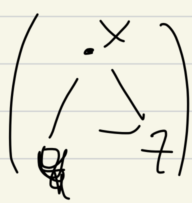
come from G



$$V_H \xrightarrow{\pi} V_G$$

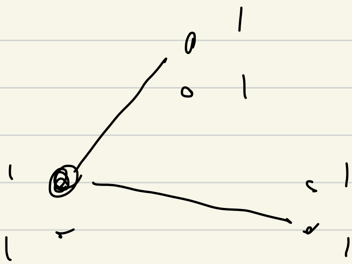
$$\{\text{even functions}\} = \left\{\begin{array}{c}\text{any function}\\ \text{on } G\end{array}\right\}\pi$$

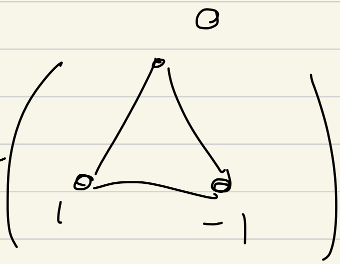
~~Fors~~ such functions:

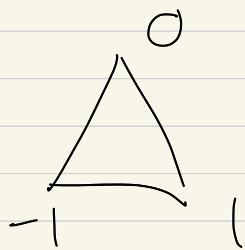
A_H  $=$ A_G  π

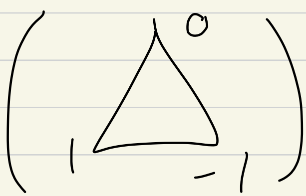
So if

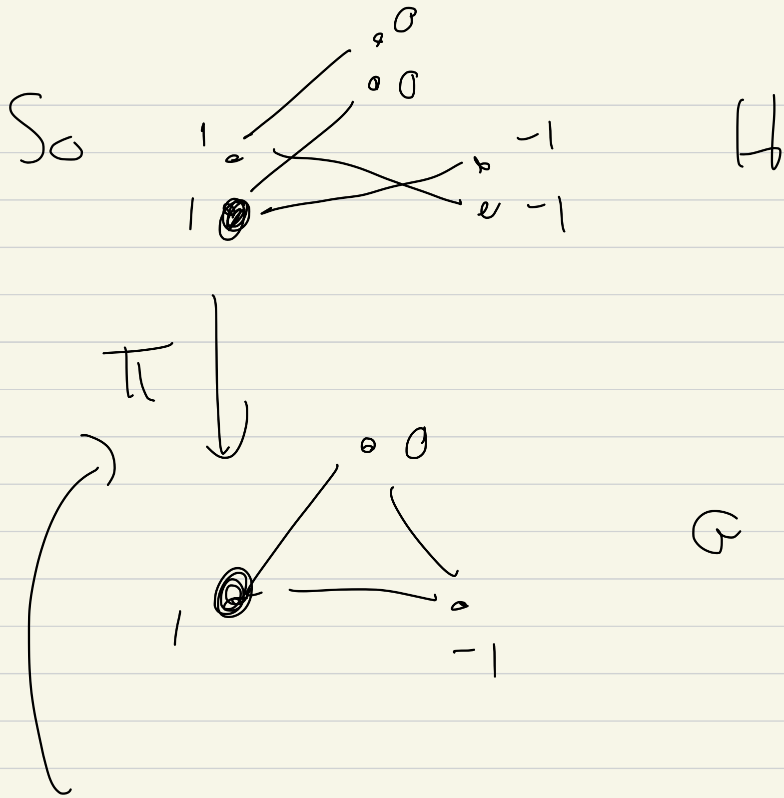
$$A_G \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$



A_G 

$=$ 

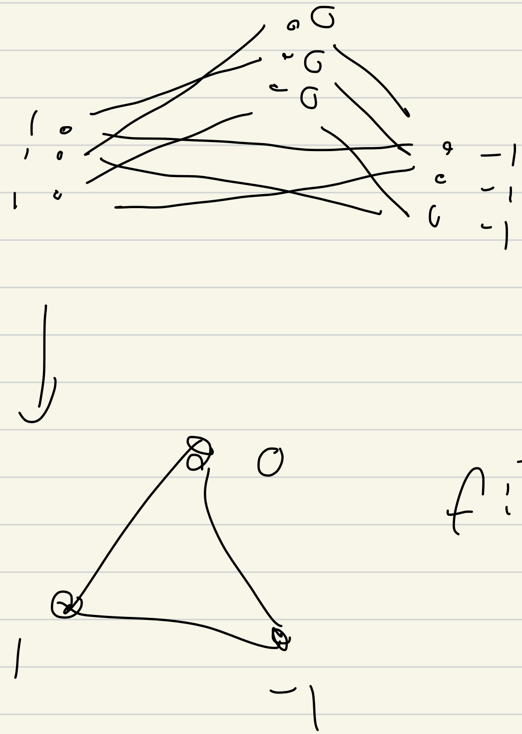
$= -1$ 



π is covering map, vertices of H see same local picture as π of the vertex or G

Rem: If $\pi: H \to G$ is

any covering map:



$f: V_G \to \mathbb{R}$

s.t. $A_G f = \lambda f$

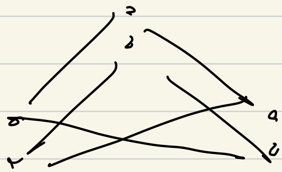
then

$$A_H (f\pi) = \lambda (f\pi).$$

Good news: deg 2 covers

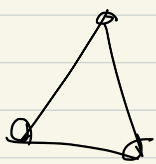
Pretty good news: "abelian covers"

Not so good news ...



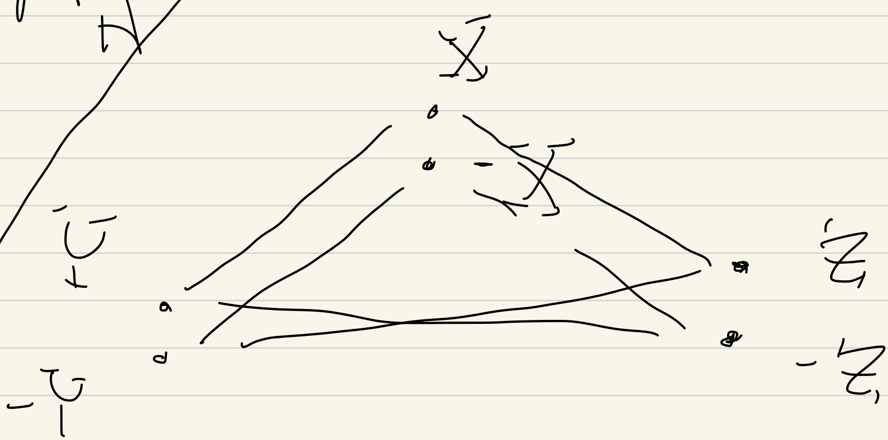
$\leftarrow$ f odd

G



$$A_G = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

A_H (



)

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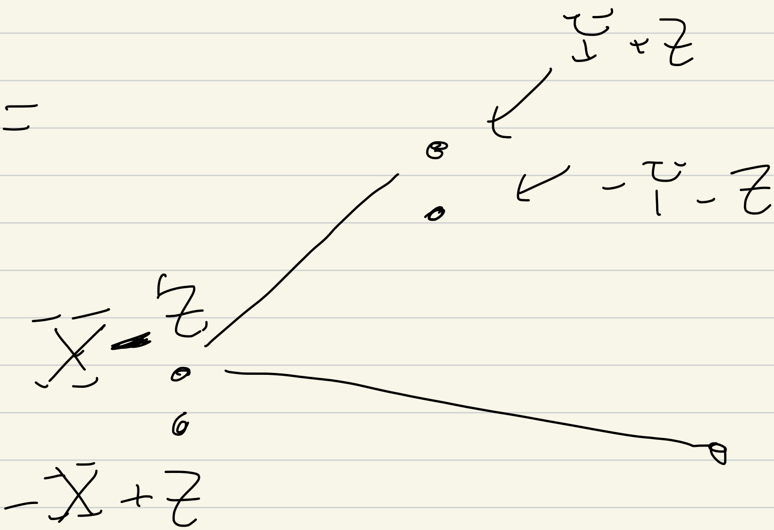


both matching are parallel

$A_H \left(\begin{array}{ccc} & X & \\ & -X & \\ Y & & Z \\ -Y & & -Z \end{array} \right) =$

$$
\begin{array}{c|ccc}
 & X & Y & Z_1 \\
\hline
X & 0 & 1 & 1 \\
Y & 1 & 0 & -1 \\
Z_1 & 1 & -1 & 0
\end{array}
= \tilde{A}_G
$$

really

$$
\left.
\begin{array}{lccc}
 & & \begin{smallmatrix} 0 & 1 \\ 0 & -1 \end{smallmatrix} & \\
\text{odd}_1 & \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} & & \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \\
\hline
 & & \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} & \\
\text{odd}_2 & \begin{smallmatrix} 1 \\ -1 \end{smallmatrix} & & \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \\
\hline
 & & \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} & \\
\text{odd}_3 & \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} & & \begin{smallmatrix} 1 \\ -1 \end{smallmatrix}
\end{array}
\right\} \text{basis for odd}
$$

Hence eigenpairs of A_H

$\Longleftrightarrow$

eigenpairs of even : $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = A_G$

+

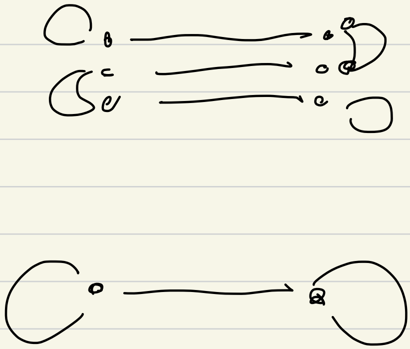
" " ~~odd~~ odd $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} = \tilde{A}_G$

signed (± 1) adjacency matrix!

1 for parallel edges

-1 " crossed "

Next time



$$\begin{tikzcd} H \arrow[d, "\pi \text{ covering map}"] \\ G \end{tikzcd}$$

But $\mathrm{Aut}_G(H) = \{id\}$