CPSC 536F March 17

For now

$$\begin{array}{ll} H & (V_H, E^{dir}_H, h_H, t_H, \cancel{\iota_H}) \\ \pi \downarrow & \\ G & (V_G, E^{dir}_G, h_G, t_G, \cancel{\iota_G}) \end{array}$$

Graph homomorphism (morphism of graphs)

=

First work with directed graphs.

Formally H, G directed graphs

a morphism $\pi: H \to G$

is really $\pi = (\pi_V, \pi_E \quad)$

where

$$\pi_V : V_H \to V_G$$

$$\pi_E : E_H^{dir} \to E_G^{dir}$$

s.t.

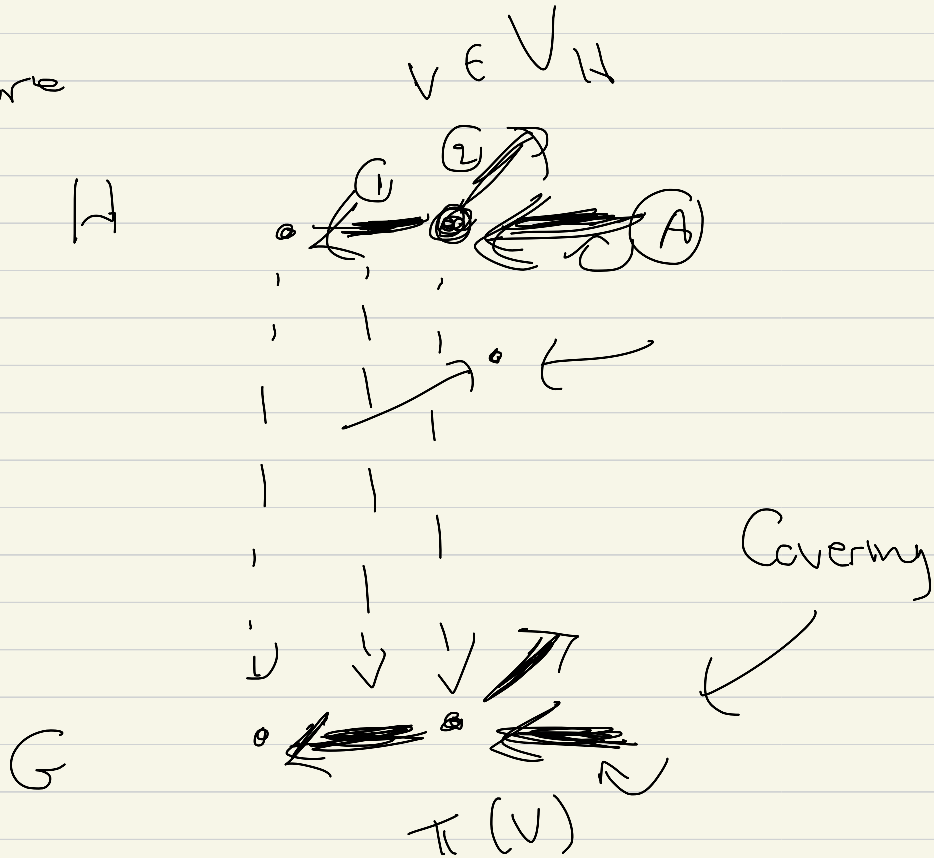
$$\begin{tikzcd}
E_H^{dir} \arrow[r, "\pi = \pi_E"] \arrow[d, "h_H"'] \arrow[dr, bend left=60, "h_G \pi_E"] \arrow[dr, bend right=80, "\pi_V h_H"'] & E_G^{dir} \arrow[d, "h_G"] \\
V_H \arrow[r, "\pi = \pi_V"'] & V_G
\end{tikzcd}$$

$$\boxed{\pi_V h_H = h_G \pi_E}$$

and

$$\boxed{\pi_V t_G = t_G \pi_E}$$

Picture



$$\text{Neighbourhood}(v) \xrightarrow{\pi} \text{Neighbourhood}(\pi(v))$$

should be a bijection!

i.e.

for any $v \in V_H$:

$$t_H^{-1}(v) = \{ e \in E_H^{dir} \mid t(e) = v \}$$

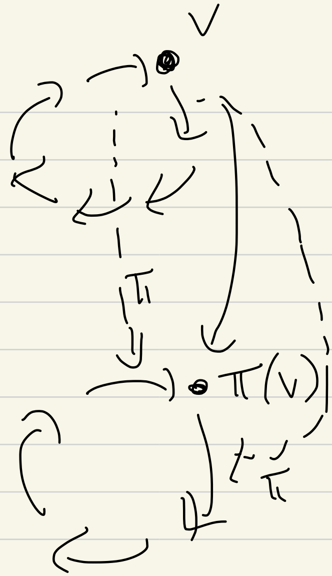
$$\begin{tikzcd} t_H^{-1}(v) \arrow[d, "\pi_E"'] \\ t_G^{-1}(\pi(v)) \end{tikzcd} \quad \text{is a bijection}$$

and

$$\begin{tikzcd} h_H^{-1}(v) \arrow[d, "\pi_E"] \\ h_G^{-1}(v) \end{tikzcd} \quad \text{is a bijection}$$

Such a π is called a covering map.

E.g. Cycle length m



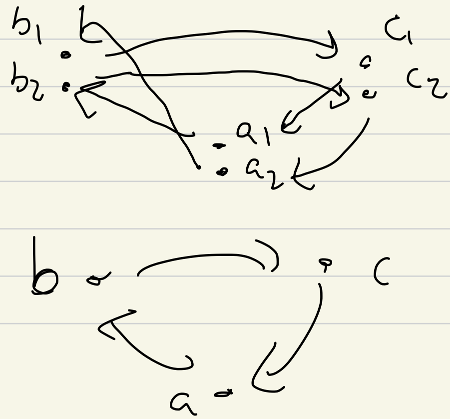
Cycle length n

there is a covering map

$$\pi : \text{Cycle}_m \to \text{Cycle}_n$$

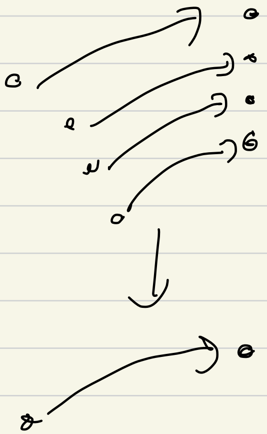
iff m is a multiple of n

e.g.



cover

If G is connected, and



$\pi^{-1}(v')$ = "the fibre of π at v'"

$v' \in V_G$

$|\pi^{-1}(v')| = k$ so

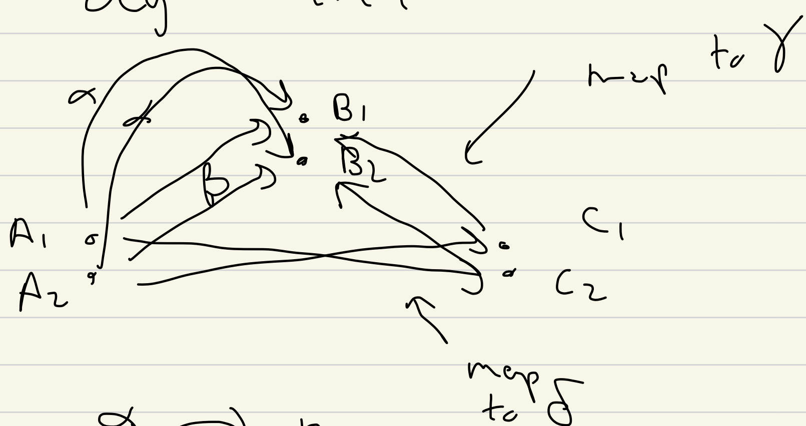
k vertices in V_H map to some v'

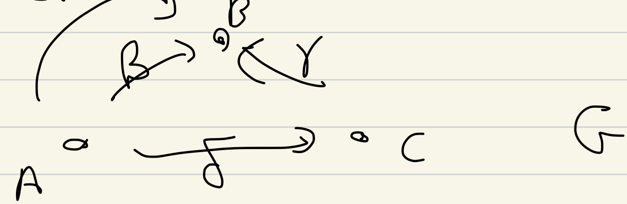
and so

k edges incident upon .

We say π is k-to-1 covering map if π^{-1} at any vertex or directed edge is of size k. We say π is "of degree k" (or a "k-lift" in CS Theory literature).

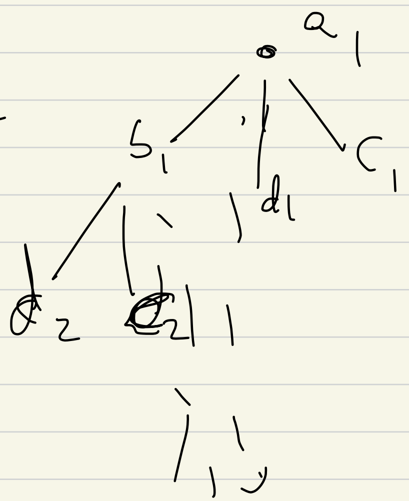
E.g. deg 2 lift:

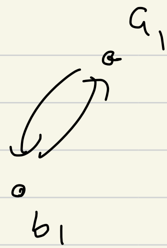




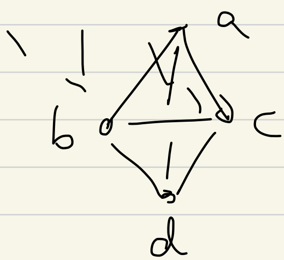
E.g.

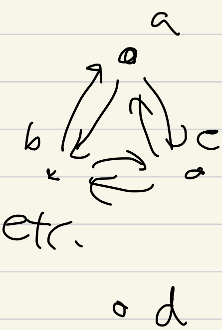
tree





3-reg graph

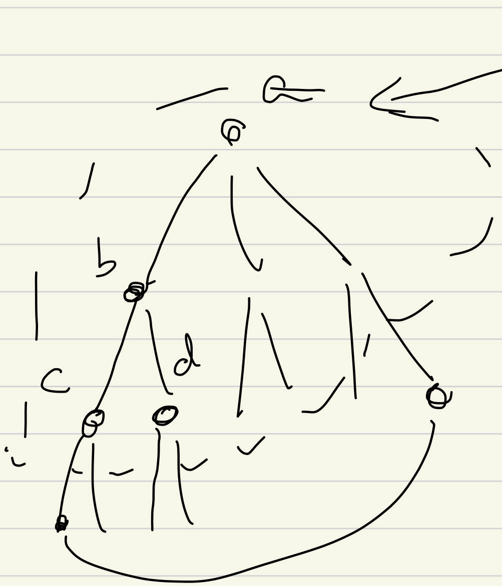


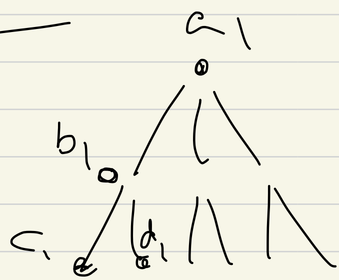


etc.

3 reg graph

building a tree





truncate

$\pi : G \to H$ is a covering map (respectively étale map) [or immersion]

if $\forall\, v \in V_H$

$$t_H^{-1}(v) \xrightarrow{\pi} t_G^{-1}(\pi(v))$$

bijection (respectively injection)

and similarly $h_H^{-1}(v) \xrightarrow{\pi} h_G^{-1}(\pi(v))$

With truncated d-regular; we want:

If $\pi : H \to G$ is étale map of finite directed graphs then

$$\lambda_1(A_H) \leq \lambda_1(A_G)$$

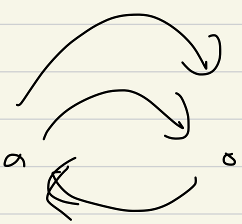
Furthermore, if $\pi : H \to G$ is covering map, then

$$\lambda_1(A_H) = \lambda_1(A_G)$$

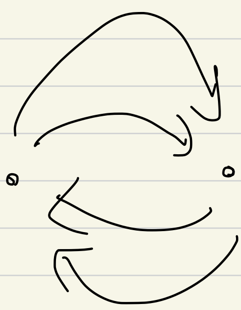
Example: If G is directed, and H is a subgraph, then

$\pi : H \longrightarrow G$ inclusion

is étale.



H

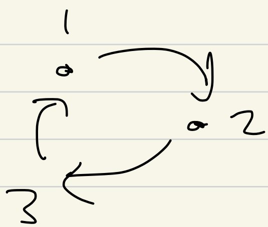


G

Example: If you take 2 permutations on $\{1, \dots, n\}$ you get a 4-regular graph

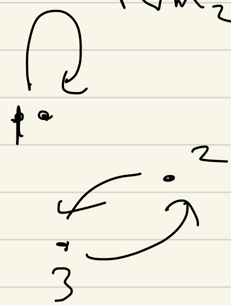
$n = 3$

Perm$_1$

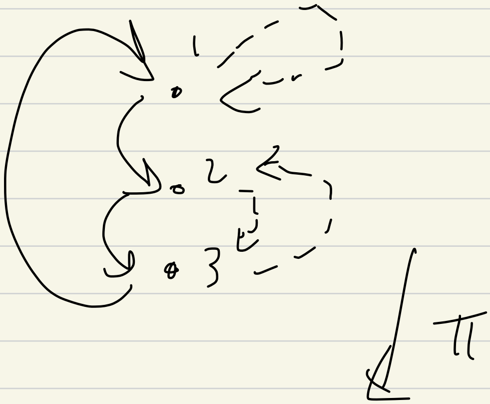


2-regular graph

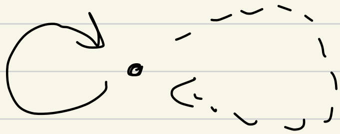
Perm$_2$



2-regular graph

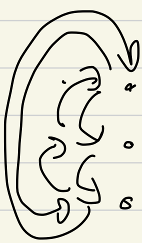


covering map



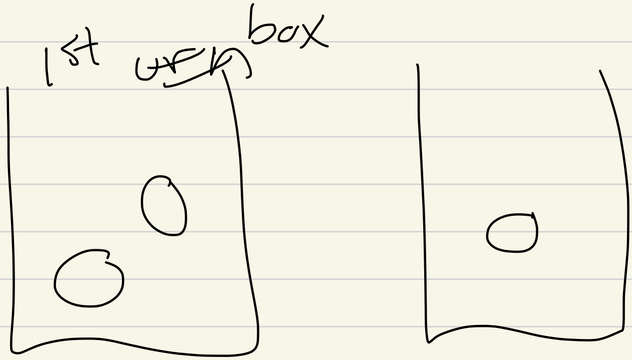
"Perm 1" "Perm 2"

reelly



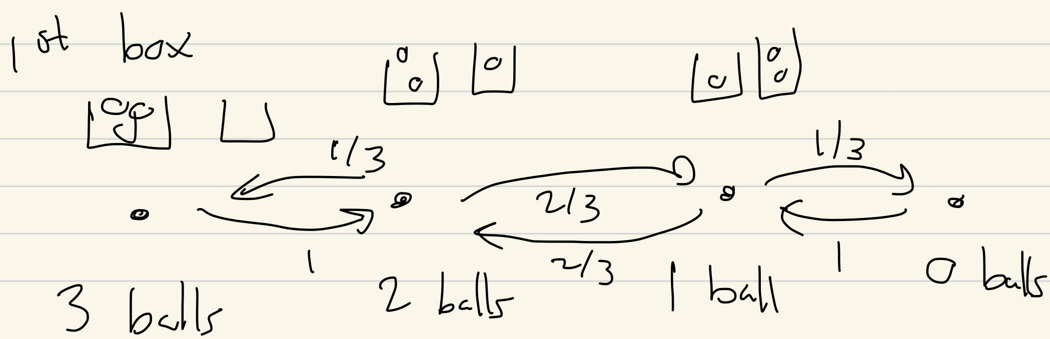
Perm 1

Example: Say you have n balls in 2 ~~urns~~ box



$n=3$

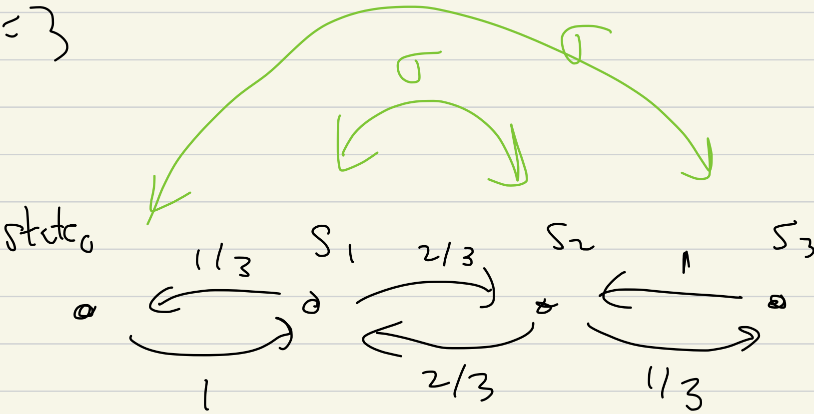
Pick one ball "at random" each with prob $1/n$, move the ball to the other ~~urn~~ box



Ehrenfest (sp.?) model

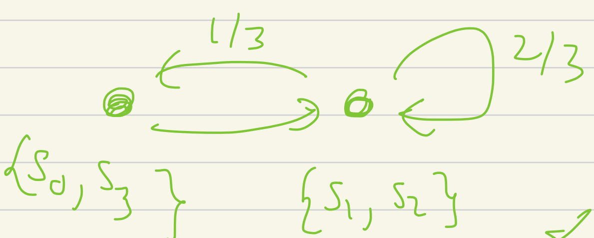
$n=3$

Markov matrix P



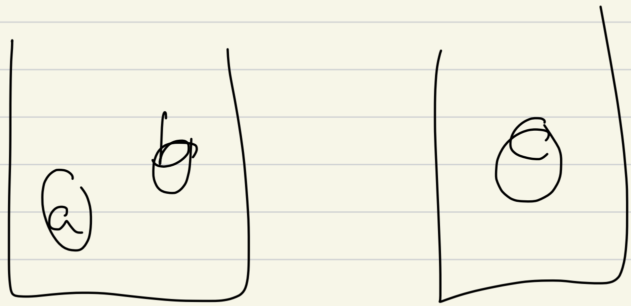
has symmetry: σ

P modulo σ

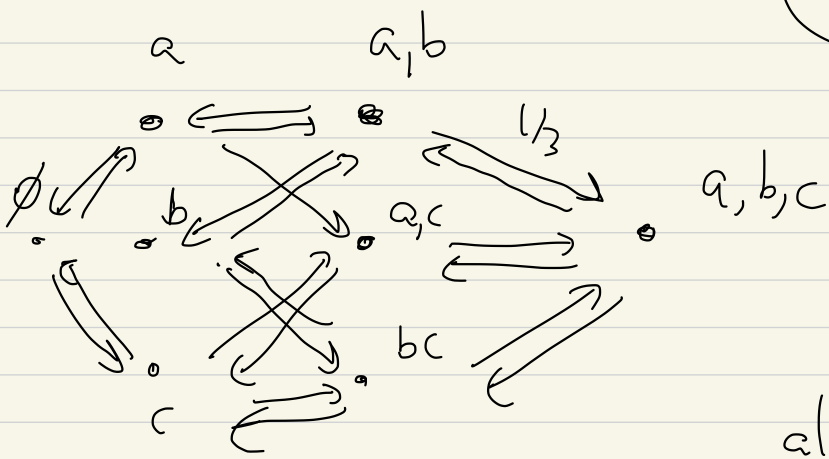


is a refinement of

Refine this model



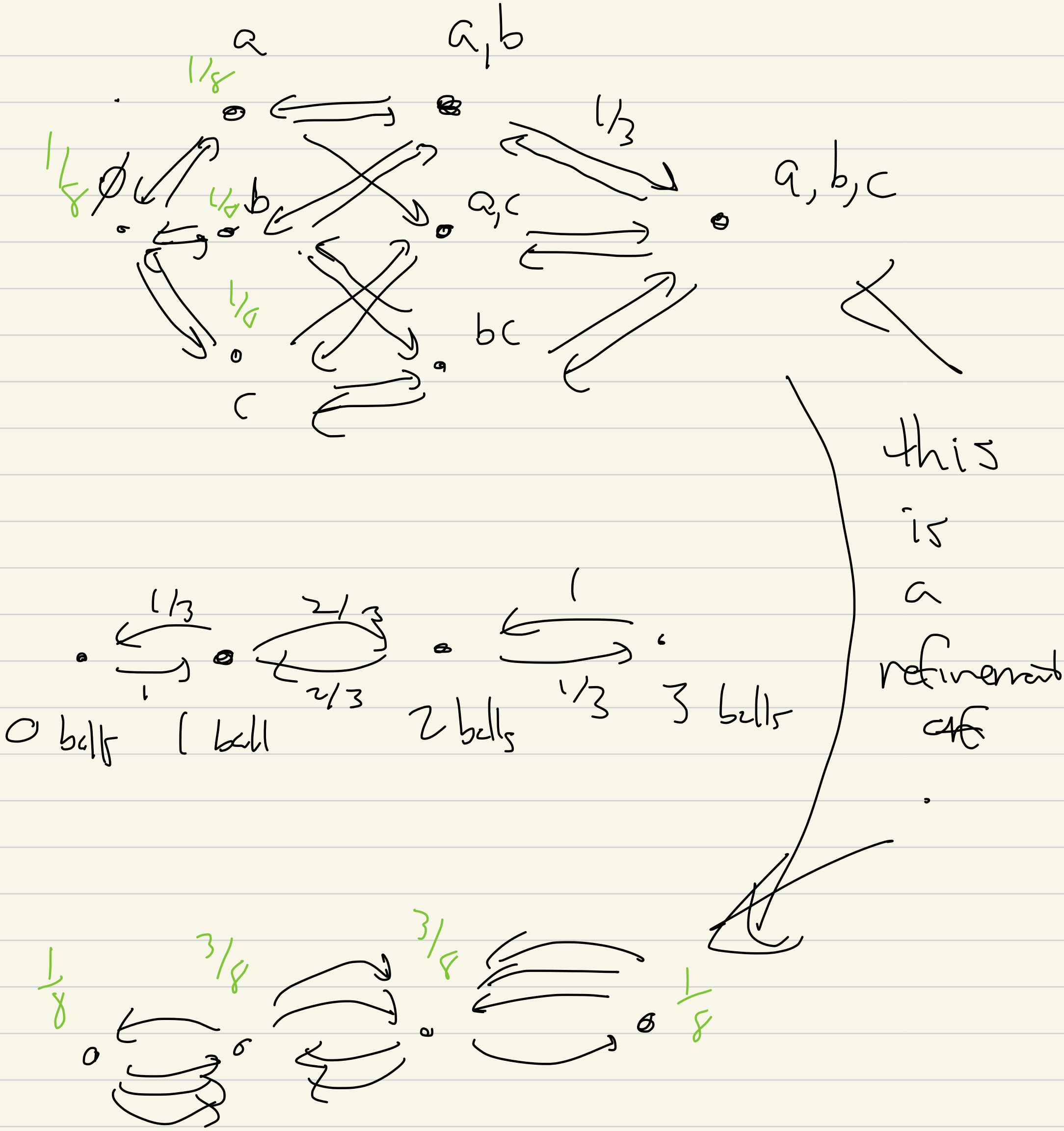
in box #1



directed graph

is $\mathbb{B}3$

all probs are 1/3



Only a covering map of digraphs