CPSC 536F March 15, 2022

Big picture!

- Let G be a graph, and

$\tilde{V}_1, \tilde{V}_2 \subset \tilde{V}_G$ s.t.

$$\text{distance}(\tilde{V}_1, \tilde{V}_2) \geq 2.$$

Let $G_1 = G|_{\tilde{V}_1}$, $G_2 = G|_{\tilde{V}_2}$

Then

$$\lambda_2(A_G) \geq \min_{i=1,2} \lambda_1(G_i)$$

If A symmetric, $n \times n$

$W < \mathbb{R}^n$, $\dim(W) = r$

$$\min_{\vec{w} \neq 0} R_A(W) \leq \lambda_r$$

Rem! If $W = \operatorname{Span}(\vec{u}_1, \ldots, \vec{u}_r)$

s.t. $A\vec{u}_i = \lambda_i \vec{u}_i$, $\vec{u}_1, \ldots, \vec{u}_r$

orthogonal, then any $\vec{w} \in W$,

$\vec{w} \neq 0 \Rightarrow \vec{w} = c_1 \vec{u}_1 + \cdots + c_r \vec{u}_r$

$$R_A(\vec{w}) = \frac{(A\vec{w}) \cdot \vec{w}}{\vec{w} \cdot \vec{w}}$$

$$= \frac{(c_1\lambda_1\vec{u}_1 + \cdots + c_r\lambda_r\vec{u}_r)\cdot(c_1\vec{u}_1 + \cdots + c_r\vec{u}_r)}{(c_1\vec{u}_1 + \cdots + c_r\vec{u}_r)\cdot(c_1\vec{u}_1 + \cdots + c_r\vec{u}_r)}$$

$$= \frac{c_1^2\lambda_1\vec{u}_1\cdot\vec{u}_1 + \cdots + c_r^2\lambda_r\vec{u}_r\cdot\vec{u}_r}{c_1^2\vec{u}_1\cdot\vec{u}_1 + \cdots + c_r^2\vec{u}_r\cdot\vec{u}_r}$$

$$\geq \frac{c_1^2\lambda_r\vec{u}_1\cdot\vec{u}_1 + \cdots + c_r^2\lambda_r\vec{u}_r\cdot\vec{u}_r}{c_1^2\vec{u}_1\cdot\vec{u}_1 + \cdots + c_r^2\vec{u}_r\cdot\vec{u}_r}$$

$$= \lambda_r.$$

So

$$\begin{pmatrix}\dim(W)\\ = r\end{pmatrix} \quad \min_{\vec{w}\in W,\ \vec{w}\neq 0} R_A(w) \leq \lambda_r$$

So

$$\max_{\dim(W)=r} \left(\min_{\substack{\vec{w} \in W \\ \vec{w} \neq 0}} R_A(w) \right) = \lambda_r$$

So "max-min principle"

Homework: Derive the min-max principle: A $n \times n$ symm., eigs $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$

$$\min_{\dim(W)=r} \left(\max_{\substack{\vec{w} \in W \\ \vec{w} \neq 0}} R_A(w) \right) = \lambda_{n-r}$$

Cor: max-min!

Say that $\vec{w}_1, \ldots, \vec{w}_r \in \mathbb{R}^n$,

A $n \times n$ symm, s.t.

(0) $\vec{w}_1, \ldots, \vec{w}_r$ non-zero

(1) $\vec{w}_i \cdot \vec{w}_j = 0$ for $i \neq j$

(2) $(A\vec{w}_i) \cdot (\vec{w}_j) = 0$ for $i \neq j$

Then $\lambda_1 \geq \ldots \geq \lambda_n$ eigenvalues of A, then

$$\lambda_r \geq \min_{i=1,\ldots,r} \left(R_A(\vec{w}_i) \right)$$

Pf: $\vec{W} = \text{Span}(\vec{w_1}, \dots, \vec{w_r})$,

then $\dim(\vec{W}) = r$.

But if $\vec{w} \in \vec{W}$, $\vec{w} \neq \vec{0}$,

then $\vec{w} = c_1 \vec{w_1} + \dots + c_r \vec{w_r}$,

So

$R_A(\vec{w})$

$$= \frac{\left(A\left(c_1\vec{w_1} + \dots + c_r\vec{w_r}\right)\right)\cdot\left(c_1\vec{w_1} + \dots + c_r\vec{w_r}\right)}{\left(c_1\vec{w_1} + \dots + c_r\vec{w_r}\right)\cdot\left(c_1\vec{w_1} + \dots + c_r\vec{w_r}\right)}$$

$$= \frac{c_1^2(A\vec{w_1})\cdot(\vec{w_1}) + \dots + c_r^2(A\vec{w_r})\cdot\vec{w_r}}{c_1^2(\vec{w_1}\cdot\vec{w_1}) + \dots + c_r^2\,\vec{w_r}\cdot\vec{w_r}}$$

We can scale $\vec{w}_1, \dots, \vec{w}_r$ s.t.

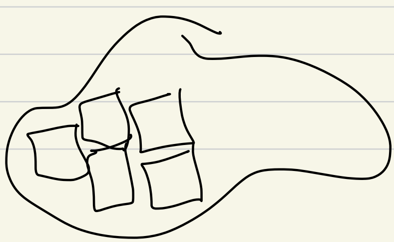
$\vec{w}_i \cdot \vec{w}_i = 1$

$$\frac{c_1^2}{c_1^2+\dots+c_r^2}\underbrace{(A\vec{w}_1)\cdot\vec{w}_1}_{= R_A(\vec{w}_1)} + \dots + \frac{c_r^2}{c_1^2+\dots+c_r^2}\underbrace{(A\vec{w}_r\cdot\vec{w}_r)}_{R_A(\vec{w}_r)}$$

(since $\vec{w}_i \cdot \vec{w}_i = 1$)

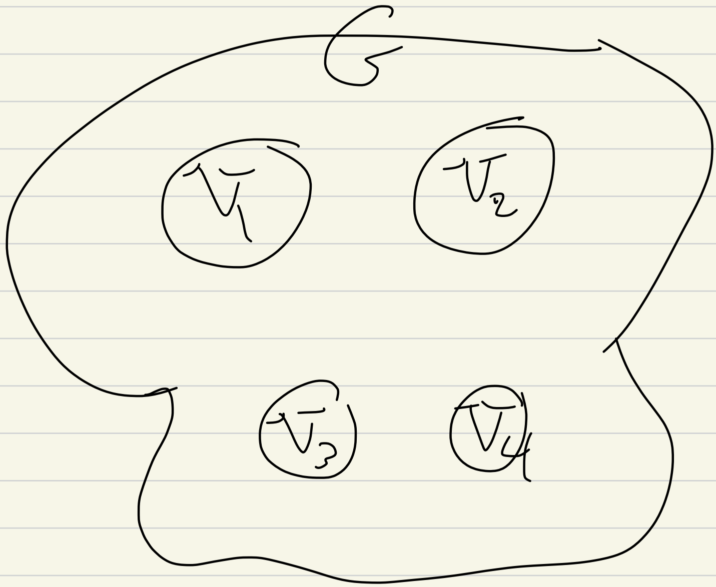
$$\geq \min\left(R_A(\vec{w}_1), \dots, R_A(\vec{w}_r)\right)$$

$\Delta u = \lambda u$



in PDE

Graph G:



Fix graph $G = (V_G, E_G, h_G, t_G, \iota_G)$

Take $V_1, \dots, V_r \subset V_G$

s.t. if $i \neq j$, $i, j \in \{1, \dots, r\}$

$\text{distance}(V_i, V_j) \geq 2$

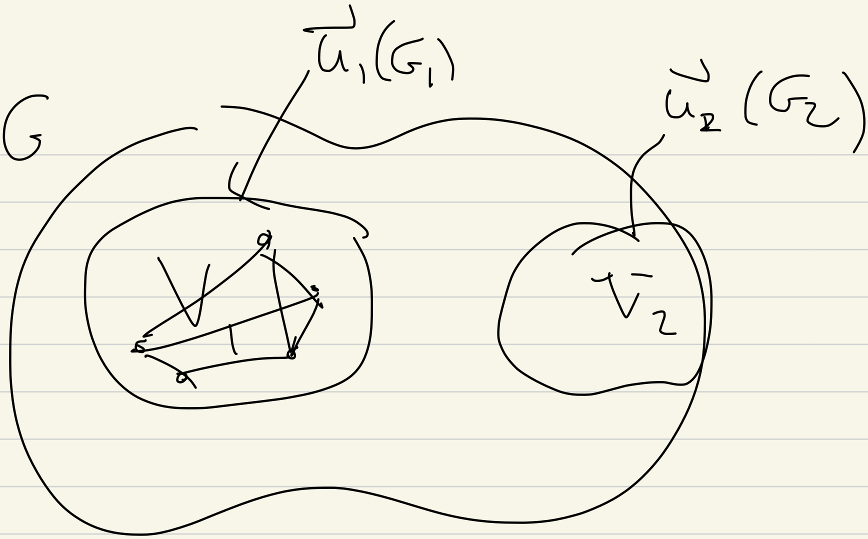
Let

G_i be $G\big|_{\bar{V}_i}$,

G_i is the "induced subgraph of G on $\bar{V}_i$"

$$V_{G_i} = \bar{V}_i$$

$$E_{G_i} = \{ e \in E_G \text{ s.t. } h(e), t(e) \in \bar{V}_i \}$$



Let eigenvalues of A_{G_i}

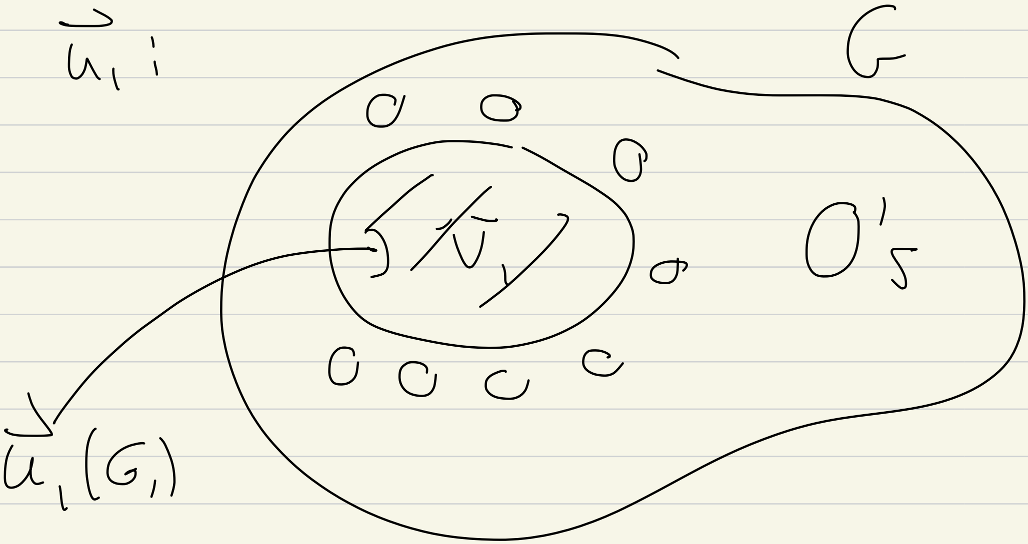
be $\lambda_1(A_{G_i})$ or $\lambda_1(G_i)$

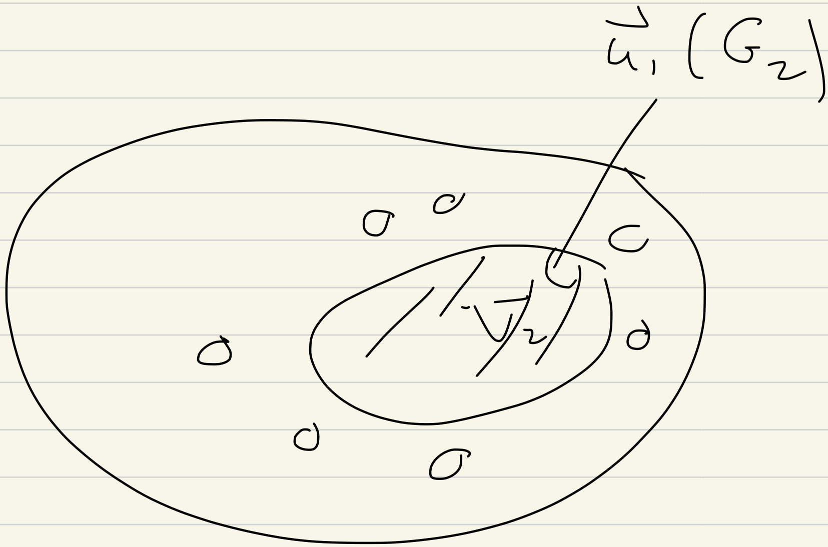
$\geq \lambda_2 \geq \dots \geq \lambda_{|V_i|}$

Thm: There is $\vec{u}_1(G_i)$ s.t.

$$R_A(\vec{u}_1(G_i)) = \lambda_1(A_{G_i})$$

$\vec{u}_1$:



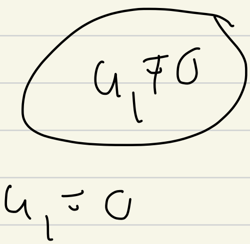


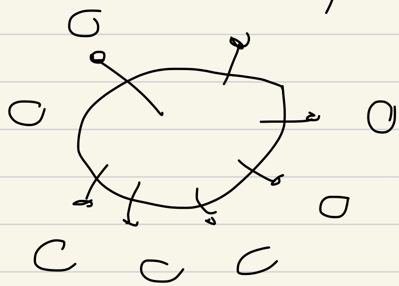
So $\vec{u}_1 \cdot \vec{u}_2 = 0$

(if $\bar{V}_1 \cap \bar{V}_2 = 0$)

Also $A\vec{u}_1(G_1)$

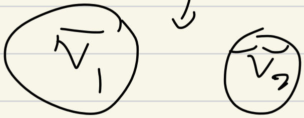
$(A\vec{u}_1)(G_1)$





$A\vec{u}_1$ is non-zero only possibly on $\bar{V}_1$ or vertices of distance 1 to $\bar{V}_1$

So distance ≥ 2



then

$$\left(A\,u_1(G_1)\right)\cdot\left(u_2(G_2)\right) = 0$$

Similarly if $\bar{V}_1, \ldots, \bar{V}_r < \bar{V}_G$

s.t. $\text{distance}(\bar{V}_i, \bar{V}_j) \geq 2$

for $i \neq j$

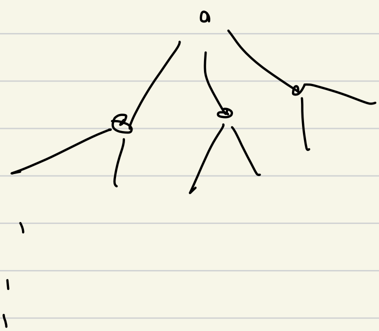
Hence

$$\lambda_r \geq \min\left(R_A(u_1(G_1)), \ldots, R_A(u_1(G_r))\right)$$

$$= \min\left(\lambda_1(G_1), \ldots, \lambda_1(G_r)\right)$$

So: G d-regular graph:

$\lambda_1(G) = d$

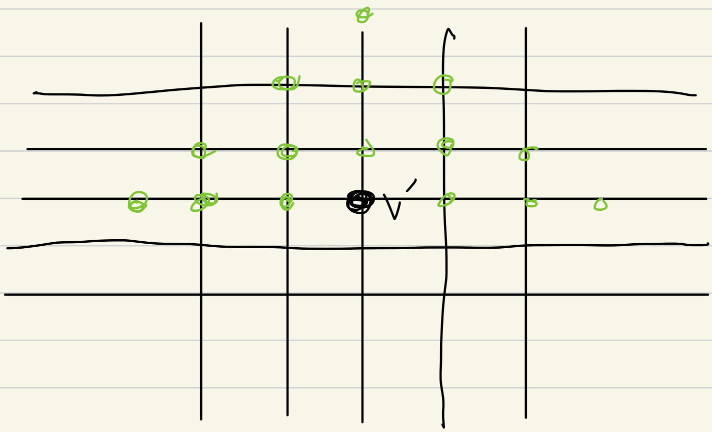


Take $v' \in V_G$, look at

$\{ v \mid \text{distance}(v, v') \leq \ell \}$

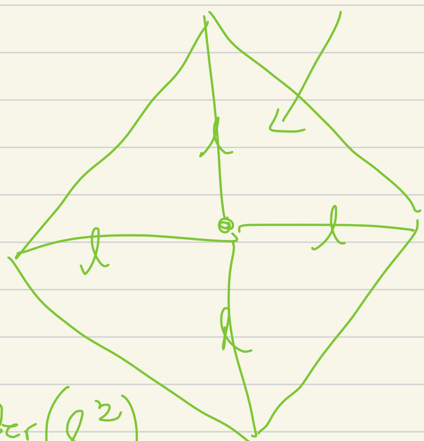
$= \text{Ball}_\ell(v')$

Grid graph $\qquad$ $\mathrm{Ball}_3(v')$



$\downarrow$

$\mathrm{Ball}_\ell(v')$



most vertices are interior

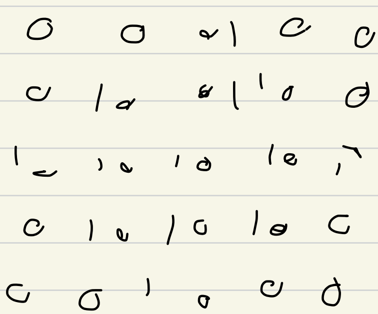
vertices = order(ℓ^2)

boundary = order(ℓ)

$\lambda_1\left(\mathrm{Ball}_\ell (v') \right)$ = roughly 4

G is $n \times m$ grid graph, $\ell < m, n$

Homework: Let $\vec{u}$ = const



$$\lambda_1\left(\Big\langle \pm \Big\rangle\right) \le 4$$

since row sums, col sums

of $A_{\mathrm{Grid}_{m,r}}\Big|_{\mathrm{Ball}_\ell(v')}$

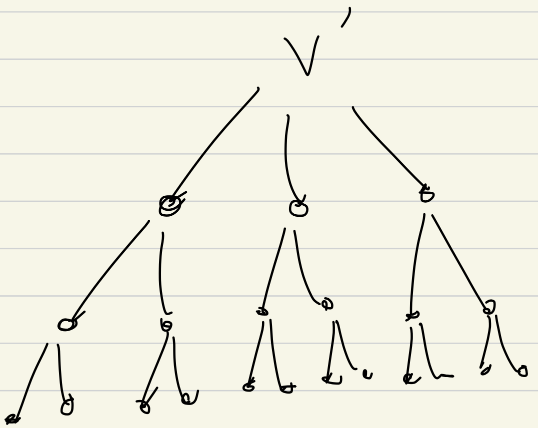
are all ≤ 4.

But $R(\text{const vector})$

$$= 4 - \mathrm{order}(1/\ell)$$

Homework.

But have a graph that looks like a tree for distances up to distance ℓ:



$d=3$

$\mathrm{Ball}_3(v')$

← a lot of vertices (most) of degree 1

Claim: $\lambda_1\left(G|_{\mathrm{Ball}_\ell(v')}\right)$

(G is d-regular graph, tree up to dist ℓ)

$$\lambda_1\left(G|_{Ball_\ell(v')}\right) \gtrsim 2\sqrt{d-1}\left(1-\frac{O(1)}{\ell^2}\right)$$

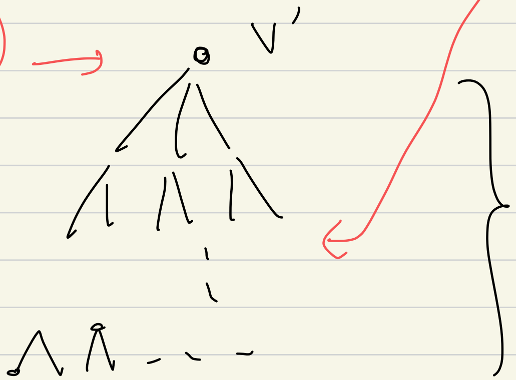
Claim: For any d-reg graph, $v' \in \tilde{V}$:

$$\lambda_1\left(G|_{Ball_\ell(v')}\right) \geq$$

$=$

Break

Say $\text{Tree}_{d\text{-reg},\ \text{size } \ell}$!



distance k level has $d(d-1)^k$ vertices

=

Give a $\vec{w}$ on $\text{Tree}_{d,\ell}$

s.t. $$\mathcal{R}_{T_{d,\ell}}(\vec{w}) = \mathcal{R}_{\text{Adjacency}_{d,\ell}}(\vec{w})$$

is close to $2\sqrt{d-1}$.

$$\mathcal{R}(\vec{w}) = \frac{(A\vec{w})\cdot(\vec{w})}{\vec{w}\cdot\vec{w}}$$

Most vertices see

distance $k-1$ to root

distance k to root

$d-1$ vert distance $k+1$ to root

Let

$$\vec{w}(v) = \left((d-1)^{-1/2}\right)^{\text{distance}(v,v')}$$

$$(A\vec{w})(v) = 2\sqrt{d-1}\,(\vec{w}(v))$$

$\vec{w}(v)\cdot\sqrt{d-1}$

$\vec{w}(v)$

$\vec{w}(v)/\sqrt{d-1}$

$d-1$ vertices

Claim: On level k $(0 < k < \ell)$

$$\underbrace{\sum_{\substack{v \text{ distance} \\ k \text{ to } v' \\ (\text{root})}} \vec{w}(v)\,\vec{w}(v)}_{} = d(d-1)^k \frac{1}{(d-1)^k}$$

$$= d$$

and

$$\underbrace{\sum (A\vec{w}(v))\,\vec{w}(v)}_{} = d(d-1)^k \frac{1}{(d-1)^k}$$

$$\cdot\, 2\sqrt{d-1}$$

Homework:

$$\sum_{k=0}^{\ell} \quad \Sigma\Big(\qquad \Big) = \text{same}\left(1 - \frac{O(1)}{\ell}\right)$$

L^2 - mass, or sum of squares

w:

1	←	level 0
d	←	level 1
d	←	level 2
$\vdots$		
d	←	level $k-1$
d	←	level ℓ

$(Aw) \cdot w$
sum level

something ≥ 0	level 0
$d \; 2\sqrt{d-1}$	level 1
$d \; 2\sqrt{d-1}$	level 2
	$\vdots$
$d \; 2\sqrt{d-1}$	level $\ell-1$
$d\sqrt{d-1}$	level ℓ

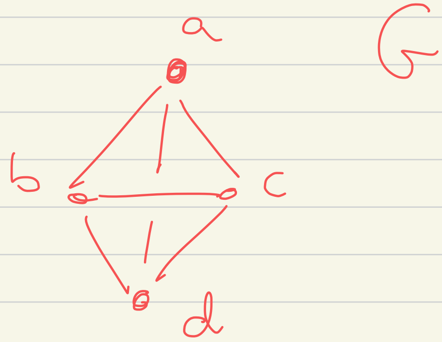
Claim: Any d-regular graph, G, $v' \in \bar{V}_G$

$$\lambda_1\left(G|_{Ball_\ell(v')}\right) \geq$$

$$\lambda_1\left(\text{Tree}_{d\text{-reg},\, \max \ell}\right)$$

$=$

Principle:



Build an infinite d-regular tree, T_d, and $T_d \to G$

Tree

look "locally"

← look locally

infinite, d-reg tree

Claim: Map

$\text{Tree} \longrightarrow \text{Graph}$

$\lambda_1(\text{Tree}) \le \lambda_1\left(\begin{array}{c}\text{image of} \\ \text{Tree}\end{array}\right)$

if this is "local" property

$\rightsquigarrow$ covering maps

étale maps