CPSC 536F March 10, 2022

Last time: really used

$$e(U,W) = \frac{d}{n}|U||W| \pm \rho\sqrt{|U||W|}$$

really took $A_G \to (A_G)^k$

$$\left| \begin{array}{l} \#\text{walks length} \\ k \text{ from } U \text{ to } W \end{array} - \frac{d^k}{n}|U||W| \right|$$

$$\leq \rho^k \sqrt{|U||W|}$$

U, W single vertizes, $|U| = |W| = 1$

$\Rightarrow$ #walks from any vertex to any other length k

$$\geq \frac{d^k}{n} - \rho^k$$

this is > 0 when

$$\frac{d^k}{n} > \rho^k$$

so

$$\frac{d^k}{\rho^k} > n \quad \text{so} \quad k > \frac{\log n}{\log(d/\rho)}$$

Gives:

diameter of G

$$\stackrel{\text{def}}{=} \max_{v_1, v_2 \in V_G} (\text{distance from } v_1 \text{ to } v_2)$$

$$\leq \frac{\log n}{\log(d/\rho)} + 1$$

Today start:

(1) If G is d-regular on n vertices, d fixed, $n \to \infty$

Alon–Boppana $\left\{ \lambda_2(G) \geq 2\sqrt{d-1} - o_n(1) \right.$

specifically

$$\lambda_2(G) \geq 2\sqrt{d-1}\left(1 - \frac{c}{(\log_{d-1} n)^2}\right)$$

(2) If $d \geq 4$, even, fixed, then most d-regular graphs on n vertices have —— for any fixed ε ⟶

$$\lambda_2(G) \leq 2\sqrt{d-1} + \varepsilon \quad [\text{Me}]$$

first result

$$\lambda_2(G) \geqslant d^{1/2} \left(2\sqrt{d-1}\right)^{1/2} + \varepsilon$$

(Broder - Shamir)

↓

Used Adjacency G

Using

$$\left\{ \begin{array}{l} \text{non-backtracking} \\ \text{Hashimoto} \end{array} \right\} \text{matrix}$$

get a better result.

⇒

Why $2\sqrt{d-1}$ ...

If G is any graph, A_G symmetric, not necessarily regular, then

$$\underbrace{\lambda_1(A_G)}_{} \geq \lambda_2 \geq \dots \geq \lambda_n(G)$$

$$\log_2(\lambda_1(A_G)) := \text{Shannon capacity of } G$$

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e.g.

Look at strings 1's, 0's s.t. no 00 occurs as a substring

"Fibonacci data"

strings length 3!

1 1 1

0 1 1   not   001

1 0 1   100

1 1 0   000

0 1 0

=

Similarly look at strings 0, 1's

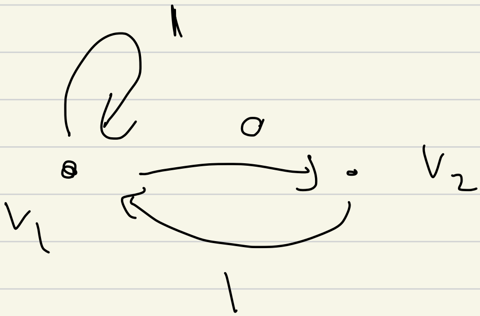
s.t. between two 1's,

at least d 0's occur

at most k 0's occur

"(d,k)-constrained data"

Fibaracci data:



G_{fib}

all Fibaracci data can be obtained by walks on this graph

start at v_1 end v_1 / v_2

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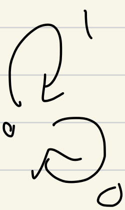


$Adj_G = [2]$

← models strngs 0,1 unconstrained

Shannon's theory:

Say you have a directed graph

$\#$ words ~~of~~ length k $\sim$ $(\lambda_1)^k$

hiding constants depend on graph

Let's make this precise ...

Thm: Say that digraph G that is strongly connected i.e. for any $v_1, v_2 \in G$

there is a path from v_1 to v_2

[Mirrors the notion of an irreducible

Markov chain.]

Then there is a single largest positive

eigenvalue λ_1 in absolute value,

and all other eigenvalues have

absolute value $\leq \lambda_1$.

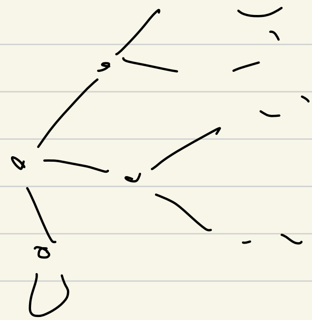
If G is a graph:

$$\lambda_1(A_G) \geq \cdots \geq \lambda_n(A_G)$$

positive

no smaller than $-\lambda_1(A_G)$

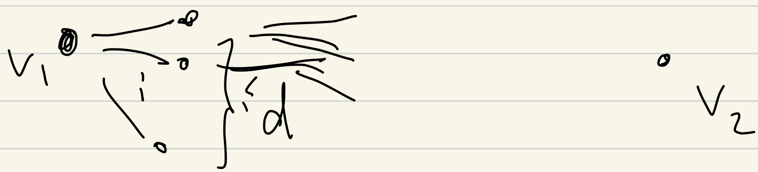
Take any d-regular graph G



Want to show $\lambda_2(G) \geq 2\sqrt{d-1}$ + small

Strategy:

take any vertex $v_1 \in V_G$. Take a v_2 s.t. distance v_1 to v_2 is as large as possible.



$\leq d(d-1)$

vertices dist 1 to v_1 $\leq d$

" " 2 to v_1 $\leq d(d-1)$

3 to v_1 $\leq d(d-1)^2$

if

$$f(k) = 1 + d + d(d-1) + d(d-1)^2 + \dots + d(d-1)^k$$

$< n$

then some vertex has distance

$> k$.

So $f(k) = d(d-1)^k \left(1 + \frac{1}{d-1} + \dots + \frac{1}{(d-1)^{k-1}} \right)$

$+ 1 + d$

So if $d \geq 3$, $1 + \frac{1}{d-1} + \frac{1}{(d-1)^2} + \dots \leq 2$

$$d(d-1)^k \, 2 < n - d - 1$$

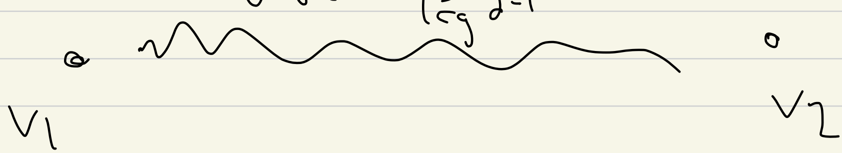
there is a v_2 of distance $k+1$ to v_1.

Or take $$k \geq \frac{\log\left(\frac{n-d-1}{2d}\right)}{\log(d-1)} - 1$$

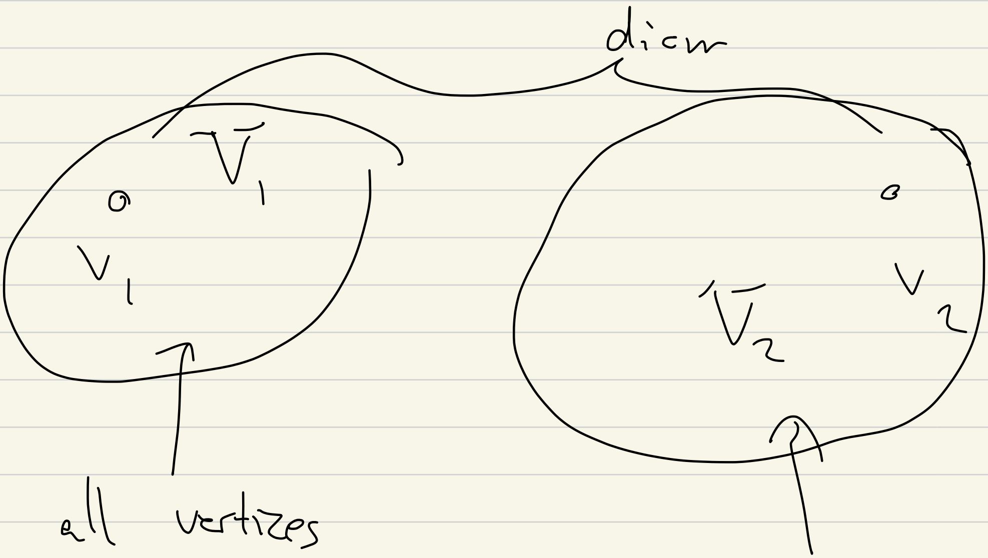
$$\geq \frac{\log n}{\log(d-1)} - c'$$

So

diam = dist $\frac{\log n}{\log d-1} - c'$



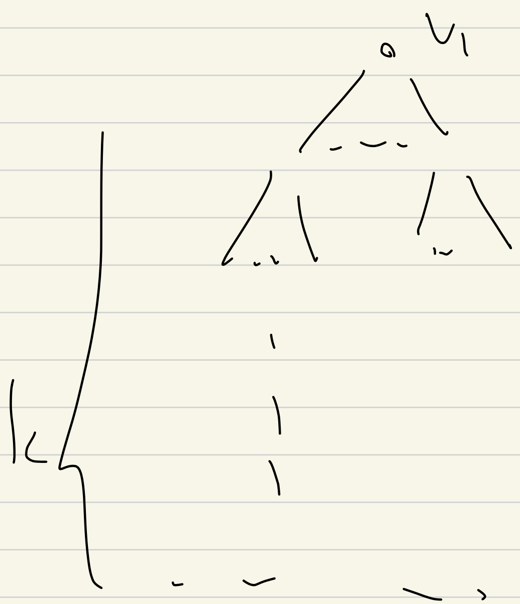
If diam = diameter of G



all vertizes of distance $\le \frac{\text{diam}}{2} - 3$

$\bar{V}_i$ = set of vertices of distance $\le \frac{\text{diam}}{2} - 3$ from v_i

Claim: d-regular graph G, and any subgraph of vertices of distance k from a $v_1 \in \overline{V}$



← at most d

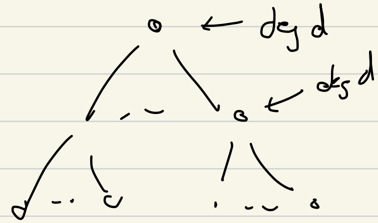
← at most $d(d-1)$ dist 2

Then

$$\lambda_1\left(\begin{array}{c}\text{Induced subgraph} \\ \text{on } \overline{V}_1 \text{ by} \\ G\end{array}\right) \geq 2\sqrt{d-1} - \underbrace{\text{function}(k)}_{\to 0 \text{ as } k \to \infty}$$

e.g. if G is locally a tree

$k=2$



← d vert

$d(d-1)$ vert

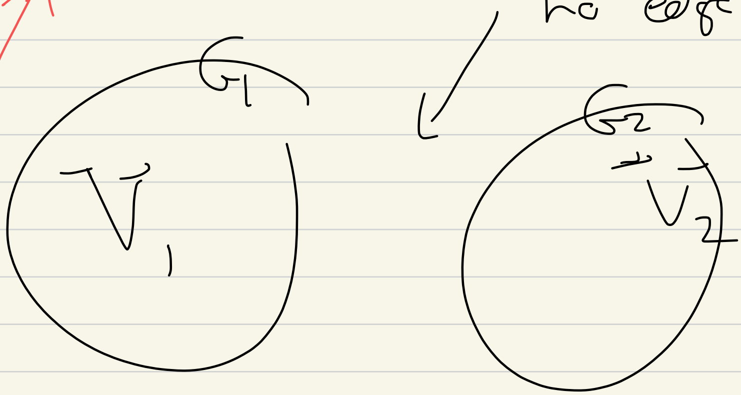
bottom: deg could be 1.

Thm: If G_1 is the induced graph by G taking a subset $\tilde{V}_1$ as above,

$$\lambda_1(G_1) \geq 2\sqrt{d-1} + o_k(1)$$

Thm: If you have

no edge from $\tilde{V}_1$ to $\tilde{V}_2$

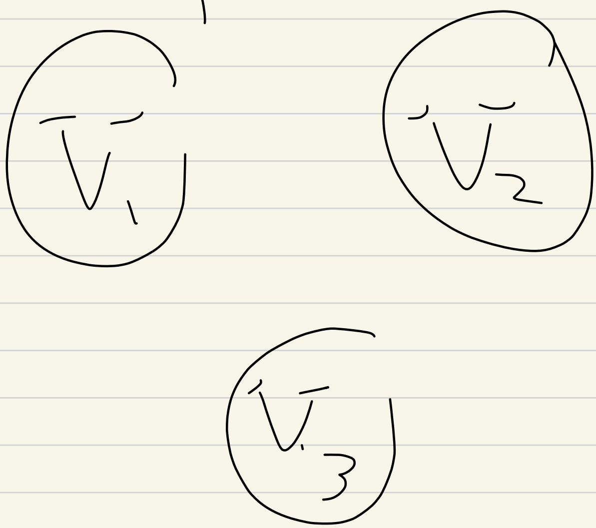


Idea behind Weyl's law, max-min, min-max

Then

$$\lambda_2(G) \geq \min\left(\lambda_1(G_1), \lambda_1(G_2)\right)$$

= Similarly



$$\lambda_3(G) \geq \min\left(\lambda_1(G_1), \lambda_1(G_2), \lambda_1(G_3)\right)$$

$=$

Min-max & max-min principles!

Say you have symmetric $n \times n$ matrix, A, eigenvalues

$\lambda_1, \ldots, \lambda_n$

Rayleigh quotient: $\vec{v} \neq 0$, $\vec{v} \in \mathbb{R}^n$

$$R(\vec{v}) = \frac{(A\vec{v}) \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$$

Rem: If $\alpha \neq 0$, $\alpha \in \mathbb{R}$

$$R(\alpha\vec{v}) = \frac{(A\vec{v}\alpha) \cdot (\vec{v}\alpha)}{(\vec{v}\alpha) \cdot \vec{v}\alpha} = R(\vec{v})$$

Let $\vec{v}_1, \ldots, \vec{v}_n$ be orthonormal eigenbasis, $A\vec{v}_i = \lambda_i \vec{v}_i$

Then if $\vec{v} \in \mathbb{R}^n$

$$\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_n \vec{v}_n$$

where

$$c_i = \vec{v} \cdot \vec{v}_i$$

$$\vec{v} \cdot \vec{v} = (c_1 \vec{v}_1 + \ldots + c_n \vec{v}_n) \cdot (c_1 \vec{v}_1 + \ldots + c_n \vec{v}_n)$$

$$= c_1^2 + c_2^2 + \ldots + c_n^2$$

(Pythagoras' law)

$$R(\vec{v}) = \frac{\left(A(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n)\right)\cdot(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n)}{\vec{v}\cdot\vec{v}}$$

$$= \frac{(c_1\vec{v}_1\lambda_1 + \cdots + c_n\vec{v}_n\lambda_n)\cdot(\qquad)}{c_1^2 + c_2^2 + \cdots + c_n^2}$$

$$= \frac{\lambda_1 c_1^2 + \lambda_2 c_2^2 + \cdots + \lambda_n c_n^2}{c_1^2 + \cdots + c_n^2}$$

$$= \lambda_1\left(\frac{c_1^2}{c_1^2 + \cdots + c_n^2}\right) + \lambda_2\left(\frac{c_2^2}{c_1^2 + \cdots + c_n^2}\right) + \cdots$$

For any $\vec{v} \neq 0$

Rem: $R(\vec{v}) \leq \lambda_1$,

Since

non-neg reals, sum 1

$$R(\vec{v}) = \lambda_1 (\quad) + \dots + \lambda_n (\quad)$$

$$\leq \lambda_1 (\quad) + \dots + \lambda_1 (\quad)$$

$$= \lambda_1$$

and $R(\vec{v}_1) = \lambda_1$

If $W \subset \mathbb{R}^n$, $\dim(W) = 2$,

then some $\vec{w} \in W$, $\vec{w} \neq 0$ s.t.

$R(\vec{W}) \leq \lambda_2.$

Why: $\vec{W} = \text{span}(\vec{w}_1, \vec{w}_2)$

Claim: There is a non-zero

α_1, α_2 s.t.

$$\left(\alpha_1 \vec{w}_1 + \alpha_2 \vec{w}_2\right) \cdot \vec{v}_1 = 0$$

Why?

$$\alpha_1 \left(\vec{w}_1 \cdot \vec{v}_1\right) + \alpha_2 \left(\vec{w}_2 \cdot \vec{v}_1\right) = 0$$

Fact: Any linear system on
m variables and $m-1$ equations has

a non-trivial solution.

So there are α_1, α_2, not all 0

s.t.

$$\vec{w} = \alpha_1 \vec{w}_1 + \alpha_2 \vec{w}_2 \neq \vec{0}$$

is orthogonal to $\vec{v}_1$. So

$$\vec{w} = \underbrace{c_1 \vec{v}_1}\, + c_2 \vec{v}_2 + \dots + c_n \vec{v}_n$$

$$c_1 = \vec{w} \cdot \vec{v}_1 = 0$$

Hence

$$R(\vec{w}) = \lambda_1 \left(\frac{c_1^2}{\sum c_i^2} \right) + \lambda_2 \left(\frac{c_2^2}{\sum c_i^2} \right) + \dots$$

$$= \lambda_2 \left(\frac{c_2^2}{\sum_i c_i^2} \right) + \ldots + \lambda_n \left(\frac{c_n^2}{\sum c_i^2} \right)$$

$$\leq \lambda_2 (\quad) + \ldots + \lambda_2 (\quad)$$

$$\leq \lambda_2 .$$

So $\dim(W) = 2$ (or ≥ 2)

then some $\vec{w} \in W$, $\vec{w} \neq 0$ has

$\vec{w} \perp \vec{v}_1$, hence

$$R(\vec{w}) \leq \lambda_2$$

Gives lower bound on λ_2.

Also, if $W = \text{Span}(\vec{v}_1, \vec{v}_2)$

then $\forall \vec{w} \in W$,

$$\vec{w} = c_1 \vec{v}_1 + c_2 \vec{v}_2$$

$$R_A(w) = \frac{c_1^2 \lambda_1 + c_2^2 \lambda_2}{c_1^2 + c_2^2}$$

$$\geq \frac{c_1^2 \lambda_2 + c_2^2 \lambda_2}{c_1^2 + c_2^2} = \lambda_2$$

$$= \max_{\dim(W) = 2,\, W \subset \mathbb{R}^n} \left(\min_{\substack{\vec{w} \in W \\ \vec{w} \neq 0}} R_A(\vec{w}) \right)$$

$= \lambda_2$

max - min principle.

$\Rightarrow$

In particular $\dim(W) = 2$

$\Rightarrow$

$$\lambda_2 \geq \min_{\substack{\vec{w} \in W \\ \vec{w} \neq 0}} R(\vec{w})$$

idea