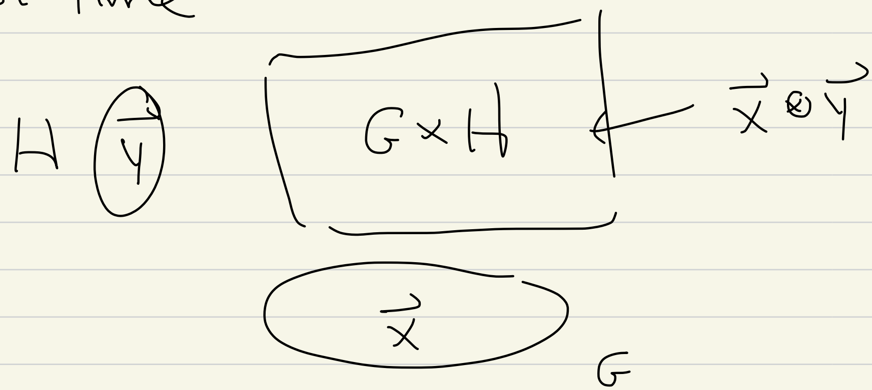
CPSC 536F March 3

Last time



$\vec{x} : V_G \to \mathbb{R}$ (complex, $\mathbb{C}$)

$\vec{y} : V_H \to \mathbb{R}$

$\vec{x} \otimes \vec{y}$ = pointwise mult

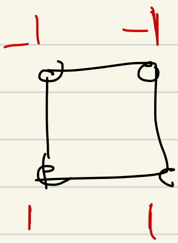
$(\vec{x} \otimes \vec{y})(u, w) = \vec{x}(u)\, \vec{y}(w)$

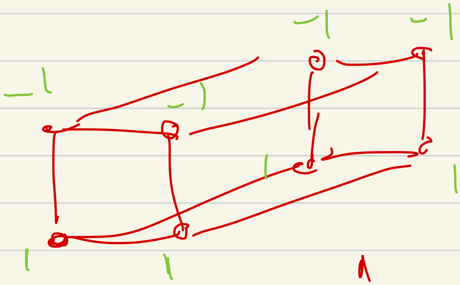
$u \in V_G,\ w \in V_H$

Example

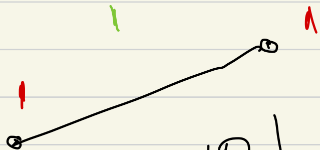
$$\mathbb{B}^3 = \mathbb{B}^1 \times \mathbb{B}^2$$

$\mathbb{B}^2$





$\vec{y}$: $A_{\mathbb{B}^2}\vec{y} = 0 \cdot \vec{y}$



$\mathbb{B}^1$

$A_{\mathbb{B}^1}\vec{x} = \vec{x}$

eigenfunctions

We say $\vec{x}_1, \dots, \vec{x}_n \in \mathbb{R}^n$

or $\vec{V}_G \to \mathbb{R}$ with $\vec{V}_G \leftrightarrow \{1, \dots, n\}$

are an <u>orthogonal basis</u>

if $\vec{x}_i \cdot \vec{x}_j = 0$ $i \neq j$,

$\vec{x}_i \neq 0$ for all i, and

<u>orthonormal</u> <u>basis</u> if

$\|\vec{x}_i\|_{L^2} = 1$.

<u>Exercise</u>: Show that if

$\vec{x}_1, \dots, \vec{x}_n$ are orthogonal basis

of $\mathbb{R}^n$, and $\vec{y}_1, \dots, \vec{y}_m$ of

$\mathbb{R}^m$, then $\{\vec{x}_i \otimes \vec{y}_j\}$ $i = 1, \dots, n$, $j = 1, \dots, m$

are an orthogonal basis for $\mathbb{R}^{n \cdot m}$. Same "orthogonal" $\longrightarrow$ "orthonormal"

=

Similarly for cartesian product of k graphs

$$G_1 \times G_2 \times \ldots \times G_k$$

=

Similarly for k^{th} Cartesian power of a graph, G:

$$G^{\times k} = \underbrace{G \times \ldots \times G}_{k \text{ times}}$$

E.g. $\mathbb{B}^k = (\mathbb{B}^1)^{\times k}$

So eigenpairs of

$\mathbb{B}^1$: A_{B^1} : $\begin{bmatrix} 1 \\ \pm 1 \end{bmatrix}$, $\lambda = \pm 1$

$\mathbb{B}^k$: $\begin{bmatrix} 1 \\ \pm 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix} \otimes \dots \otimes \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix} = \vec{x}_{\mathbb{B}^k}$

Exercise!

If $\vec{x} : V_G \to \mathbb{R}$ $\quad A_G \vec{x} = \lambda \vec{x}$

$\vec{y} : V_H \to \mathbb{R}$ $\quad A_H \vec{y} = \mu \vec{y}$

then

$$A_{G \times H} (\vec{x} \otimes \vec{y}) = (\lambda + \mu) (\vec{x} \otimes \vec{y})$$

So corresponding eigenvalue to $\vec{X}_{\mathbb{B}^k}$

is $(\pm 1) + (\pm 1) + \dots + (\pm 1)$

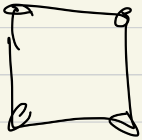
i.e.

$$\vec{X}_{\mathbb{B}^k} = \begin{bmatrix} 1 \\ s_1 \end{bmatrix} \otimes \dots \otimes \begin{bmatrix} 1 \\ s_k \end{bmatrix}$$

$$s_1, \dots, s_k \in \{\pm 1\}$$

$$\lambda = s_1 + \dots + s_k$$

$=$



$\mathbb{B}^2$:

$$\begin{bmatrix}1\\1\end{bmatrix}\otimes\begin{bmatrix}1\\1\end{bmatrix} \qquad \lambda = 2$$

$$\left.\begin{matrix}\begin{bmatrix}1\\1\end{bmatrix}\otimes\begin{bmatrix}1\\-1\end{bmatrix}, \\ \begin{bmatrix}1\\-1\end{bmatrix}\otimes\begin{bmatrix}1\\1\end{bmatrix}\end{matrix}\right\} \quad \lambda = 0$$

$$\left.\begin{bmatrix}1\\-1\end{bmatrix}\otimes\begin{bmatrix}1\\-1\end{bmatrix}\right\} \quad \lambda = -2$$

$\Rightarrow$ Similarly, λ's of $A_{\mathbb{B}^k}$ are

$$(\pm 1) + (\pm 1) + \ldots + (\pm 1)$$

$k = 1 + 1 + \dots + 1$ mult 1

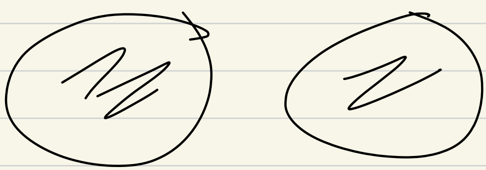
$k-2 = \underbrace{\qquad\qquad}_{\text{all 1's except one } -1}$ mult $\binom{k}{1}$

$k-4$ mult $\binom{k}{2}$

$\vdots$

$-k$ mult $\binom{k}{k} = 1$

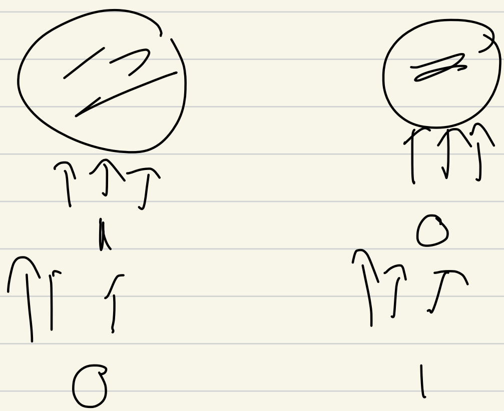
Note: d-regular 2 conn comp



d occurs with mult 2;

mult d =

$$\dim\left(\ker\left(A_G - dI\right)\right)$$



and

$\dim = 2$

$$\cdots \lambda_3 < \lambda_2 = d \leq \lambda_1 = d$$

$k-2$ eigenvalues: $\mathbb{B}^k$ $\qquad \mathbb{B}^{k-1} \qquad \mathbb{B}^{k-1}$

in $\mathbb{B}^k$

near k

$\lambda = k-2$ eigenvector of $A_{\mathbb{B}^k}$

$\vec{x} =$ 1 on the left, -1 on the right

$\mathbb{B}^{k-1}$ $\mathbb{B}^{k-1}$

i

only one edge from any vertex on left side to one on the right

every vertex has $k-1$ neighbors

$\lambda = -k$

all edges are in here

even vertices

nothing

$0 0 \ldots 0$

$1 1 0 0 \ldots 0$

$1 0 1 0 0 \ldots 0$

everything

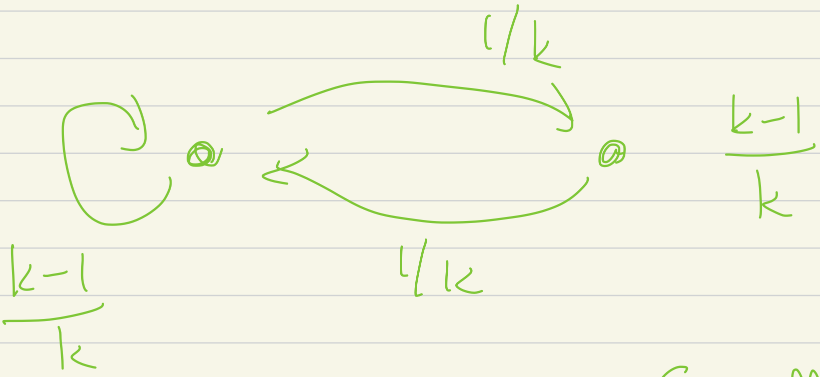
odd vertices

nothing

$1 0 0 \ldots 0$

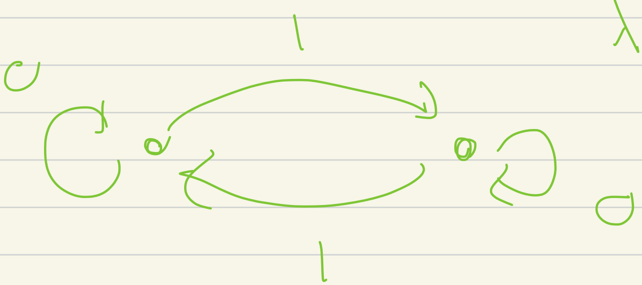
$0 1 0 \ldots 0$

Analogous to $\lambda = k-2$



Corr Markov chain

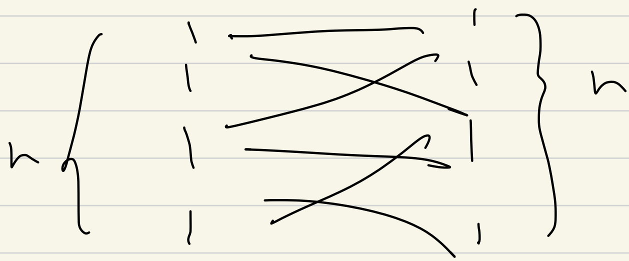
$\lambda = -k$



We'll make this precise when we get to covering maps.

Remark: We'll use this theory to get algorithms to approximate # perfect matchings in bipartite graph ....

Idea:



look at all matchings:

matching with 1 edge = # edges

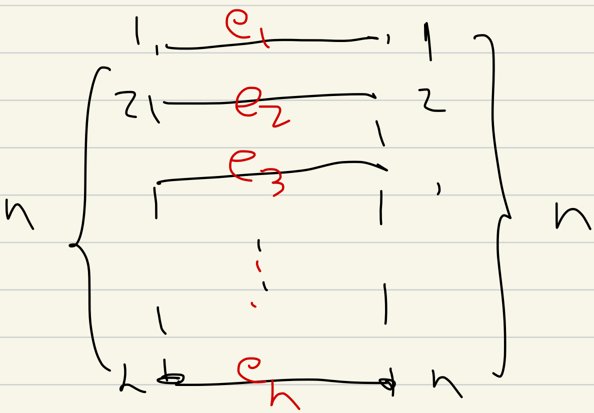
If we can approx

$$\frac{\#\text{ matching with 2 edges}}{\#\text{ match-- 1 edge}} \cdot \frac{\# \ldots 3 \ldots}{\# \ldots 2 \ldots} \cdots$$

we'll use this to get approx for

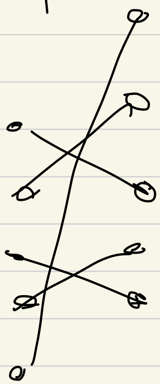
$$\frac{\#\text{ matchings } n \text{ edges}}{\#\text{ '' '' } 1 \text{ edge}}$$ ← perfect matching

Degenerate case:



$\#$ matching k edges: $\binom{n}{k}$

Approximation: We'll random walk on a "configuration graph" adjacency



add edge

perfect matching of k edges

matching of k+1 edges

delete an edge

matching k-1 edges

Consider the degenerate case!

then matchings $\longleftrightarrow$

subsets of $\{e_1, \dots, e_n\}$

adjacency

$n = 5$

$\{e_1, e_2, e_4\}$ add an edge

$e_1 e_2$

$e_1 e_4$

$e_2 e_4$

$e_1 e_2 e_3 e_4$

$e_1 e_2 e_4 e_5$

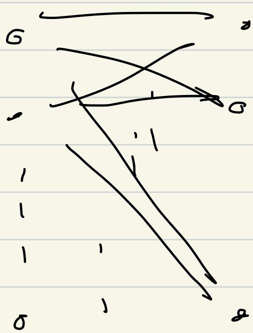
This configuration graph is just the hypercube.

$$\frac{\binom{n}{n}}{\binom{n}{1}} = \frac{\binom{n}{n}}{\binom{n}{n-1}} \cdot \frac{\binom{n}{n-1}}{\binom{n}{n-2}} \cdots \frac{\binom{n}{2}}{\binom{n}{1}}$$

$\|$

$$\frac{\#\text{ perfect matchings}}{\#\text{ matchings 1 edge}}$$

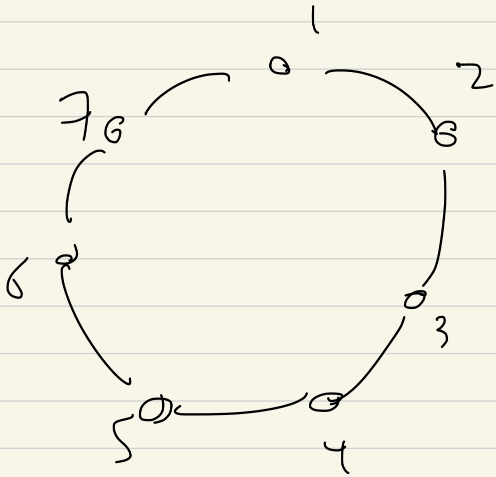
Another deg case



$K_{n,n}$ = complete bipartite graph

3 min break 10:25 - 10:28

Cycle: Cycle length n

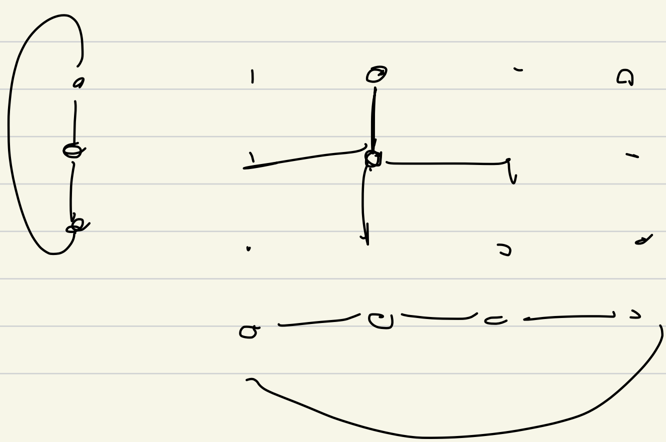


2-regular graph

bad for eigenvalues

and

"grid graph"



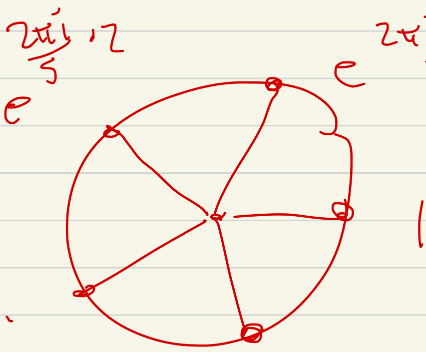
Cycles:

$$A_{\text{Cycle}_n} = \begin{bmatrix} 0 & 1 & & & 0 & 1 \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & \ddots & \ddots & 1 \\ 1 & & & & 1 & 0 \end{bmatrix}$$

Let $\zeta \in \mathbb{C}$ s.t. $\zeta^n = 1$

i.e. ζ is n-th root of unity

$e^{2\pi i/5} = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

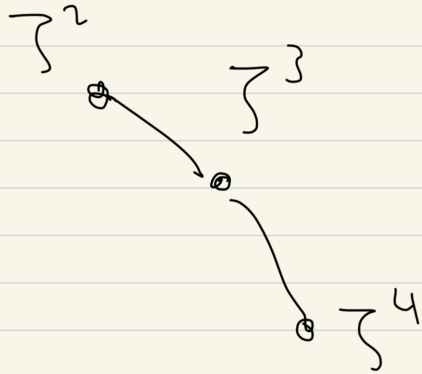


O_ζ

S_σ

$$\begin{bmatrix} 0 & 1 & & & 1 \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & & 1 \\ 1 & & & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \zeta \\ \zeta^2 \\ \vdots \\ \zeta^{n-1} \end{bmatrix} \qquad \vec{x} = \vec{x}_\zeta$$

O'_ζ



$$\left(A_{C_n} \vec{x}_\zeta\right)_3 = \zeta^2 + \zeta^4$$

$$= \zeta^3\left(\zeta^{-1} + \zeta^1\right)$$

More generally "circulant matrix"

$$\Bigg(\underbrace{\begin{bmatrix} a_0 & a_1 & a_2 & \text{etc.} & & \\ & a_0 & a_1 & \ddots & & \\ & & a_0 & & \ddots & a_2 \\ a_2 & \text{etc.} & & \ddots & & a_1 \\ a_1 & a_2 & & & & a_0 \end{bmatrix}}_{A} \begin{bmatrix} 1 \\ \zeta \\ \vdots \\ \vdots \\ \zeta^{n-1} \end{bmatrix} \Bigg)_{3^{rd}\ \text{comp}}$$

$a_{n-1}\zeta^2 = a_{-1}\zeta^2$

ζ^2

$\zeta^3 \qquad a_0\zeta^3$

$a_1\zeta^4$

$\zeta^4 \qquad a_{-1}\zeta^{-1}$

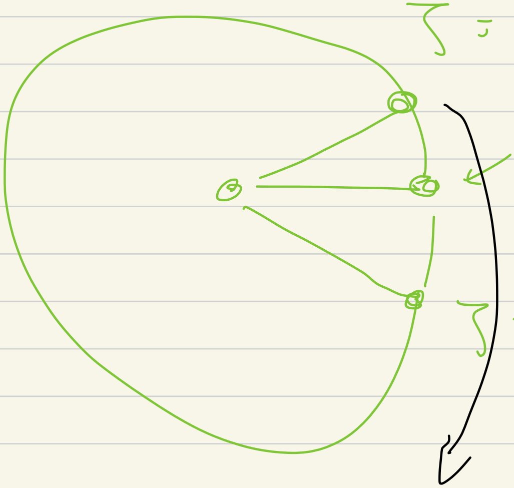
similar eigenvalue is

$a_0 + a_1\zeta + a_2\zeta^2 + \dots + a_{n-1}\zeta^{n-1}$

For cycle: eigenvalues

$$\{\zeta + \zeta^{-1}\}_{\zeta^n = 1}$$

$$\zeta = e^{\frac{2\pi i}{n}}$$



$\zeta = 1$ $\qquad$ 2

$\zeta = e^{-\frac{2\pi i}{n}}$

vector $\begin{pmatrix}1\\ \zeta \\ \vdots \\ \zeta^{n-1}\end{pmatrix} = \begin{pmatrix}1\\1\\ \vdots \\1\end{pmatrix}$

$$e^{\frac{2\pi i}{n}} + e^{-\frac{2\pi i}{n}}$$

$$\left(\cos\left(\frac{2\pi}{n}\right) + i\sin()\right) + \left(\cos\frac{2\pi}{n} - i\sin()\right)$$

$$= 2\cos\frac{2\pi}{n}$$

$$2\cos\frac{2\pi}{n}$$

roughly

$$2\left(1-\frac{1}{2}(\quad)+\frac{1}{4!}(\quad)^2-\cdots\right)$$

$$2\left(1-\frac{1}{2}\left(\frac{2\pi}{n}\right)^2+O\left(\frac{1}{n^4}\right)\right)$$

$$2-(2\pi)^2\frac{1}{n^2}+O\left(\frac{1}{n^4}\right)$$

really bad

Both $\quad \mathrm{Re}\begin{bmatrix}1\\ \zeta\\ \vdots\\ \zeta^{n-1}\end{bmatrix}$

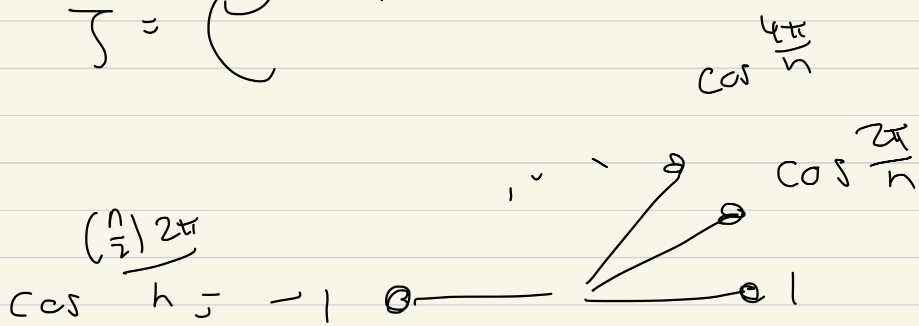
and

$\quad \mathrm{Im}\begin{bmatrix}1\\ \zeta\\ \vdots\\ \zeta^{n-1}\end{bmatrix}$

are eigenvectors

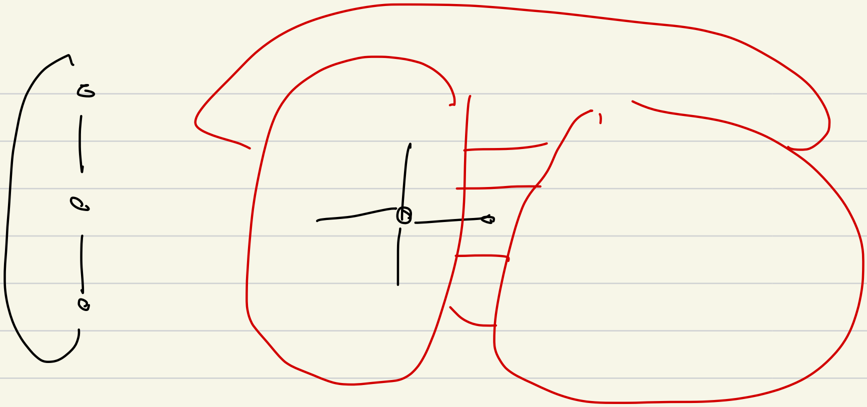
so

$$\zeta = e^{\frac{2\pi i}{n}}$$

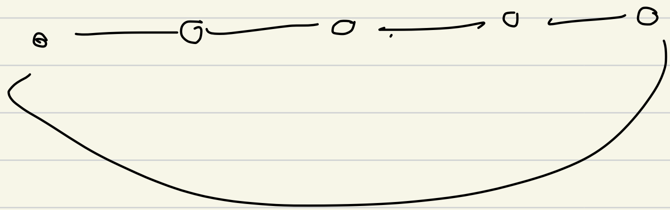


(n even)

Cycle_m



grid graph



Cycle_n

d-regular, $d = 4$

$$\lambda = \left\{ \begin{array}{l} 2, \\ 2\cos\frac{2\pi}{n}, \\ 2\cos\frac{4\pi}{n}, \\ \vdots \end{array} \right\} + \left\{ \begin{array}{l} 2, \\ 2\cos\frac{2\pi}{m} \\ \vdots \\ \vdots \end{array} \right\}$$