CPSC 536F March 1, 2022

HW #1 on circuits/formulas done.

Starting graphs & eigenvalues.

Today: Examples.

Just before: some theory:

Review what is often called

"The Expander $\underbrace{\text{Mixing}}_{\text{Markov chain mixing time}}$ Lemma"

Refer:

Markov Chains and Mixing Times,

by Levin, Peres, Wilmer

(Specifically Chapter 4 on Mixing Times)

$\Rightarrow$

Review: Say G is d-regular graph:

$$A_G = \begin{bmatrix} 0's & \\ & 1's \end{bmatrix} \begin{array}{l} \leftarrow \text{each} \\ \leftarrow \text{row} \\ \leftarrow \text{sum} \\ = d \end{array}$$

each col sum $= d$

Claim that

$A_G =$ adjacently matrix, i.e.

$(A_G)_{i,j} =$ # edges from i to j

then eigenvalues of A_G are real!

$$\lambda_n(G) \leq \ldots \leq \lambda_2(G) \leq \lambda_1(G)$$
$$\lambda_1(G) = d$$

$$A_G \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} d \\ d \\ \vdots \\ d \end{bmatrix} = d \vec{1}$$

Fact 1:

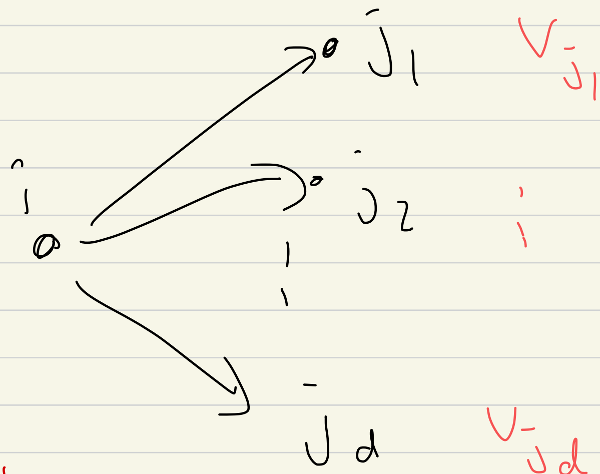
All eigenvalues λ_i have

$$|\lambda_i| \le d$$

Pf: $A\vec{v} = \lambda \vec{v}$

look at largest comp of $\vec{v}$,

say v_i



say $|v_i|$ largest abs value

$$(A\vec{v})_i \overset{\text{def}}{=} v_{j_1} + \dots + v_{j_d}$$

where edges from i run to

$j_1, \dots, j_d$

So

$$A\vec{v} = \lambda \vec{v}$$

$$(A\vec{v})_i = \lambda \vec{v}_i$$

$$|\lambda v_i| = |\lambda|\,|v_i|$$

$$= |v_{j_1} + \dots + v_{j_d}|$$

$\leq |v_{j_1}| + \dots + |v_{j_d}| \leq d\,|v_i|$

Hence $|\lambda| \leq d$.

"Maximum principle"

Refine: Say that

$|v_i| = M$ is max abs value.

Then

$$|\lambda|\, M = |v_{j_1} + \dots + v_{j_d}|$$

$$\leq d \max_{k=1,\dots,d} |v_{j_k}| \leq d\,M$$

So $|\lambda| = d$, then

ecl $|v_{j_2}| = m$

So

$$v_{j_1} = \pm m, \ldots, v_{j_d} = \pm m$$

So $|v_{j_1} + \ldots + v_{j_d}|$

$$= |v_{j_1}| + \ldots + |v_{j_d}|,$$

then

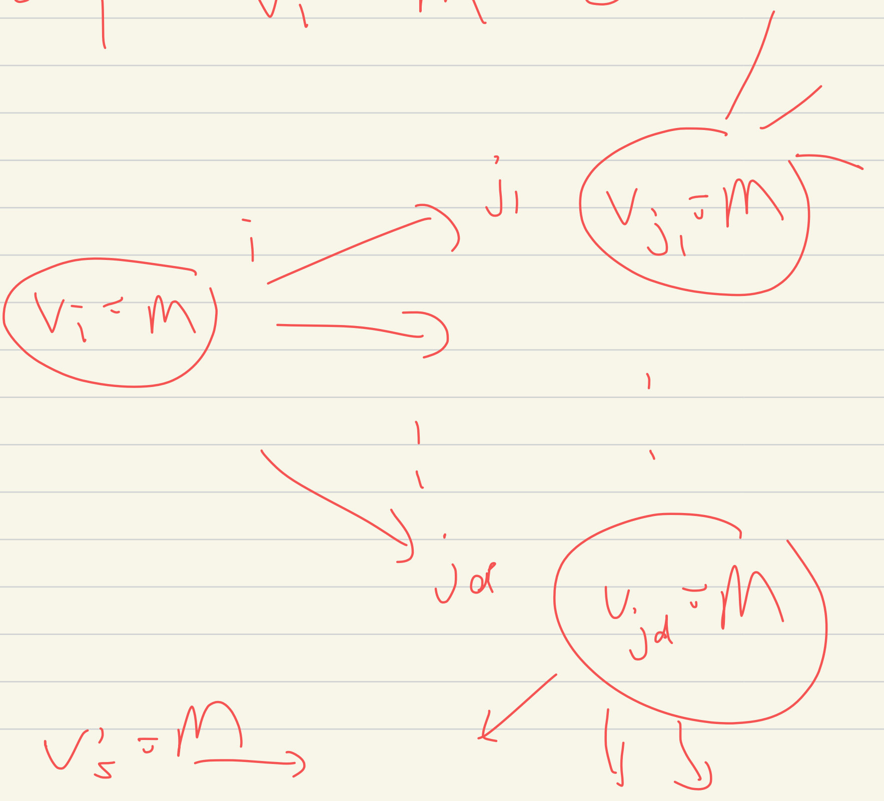
$$v_{j_1} = v_{j_2} = \ldots = v_{j_d} = \pm m$$

$\lambda = d,$

$(Av)_i = \lambda v_i$

$= v_{j_1} + \dots + v_{j_d}$ $\qquad v_j = \mu$

and say $v_i = \mu > 0$

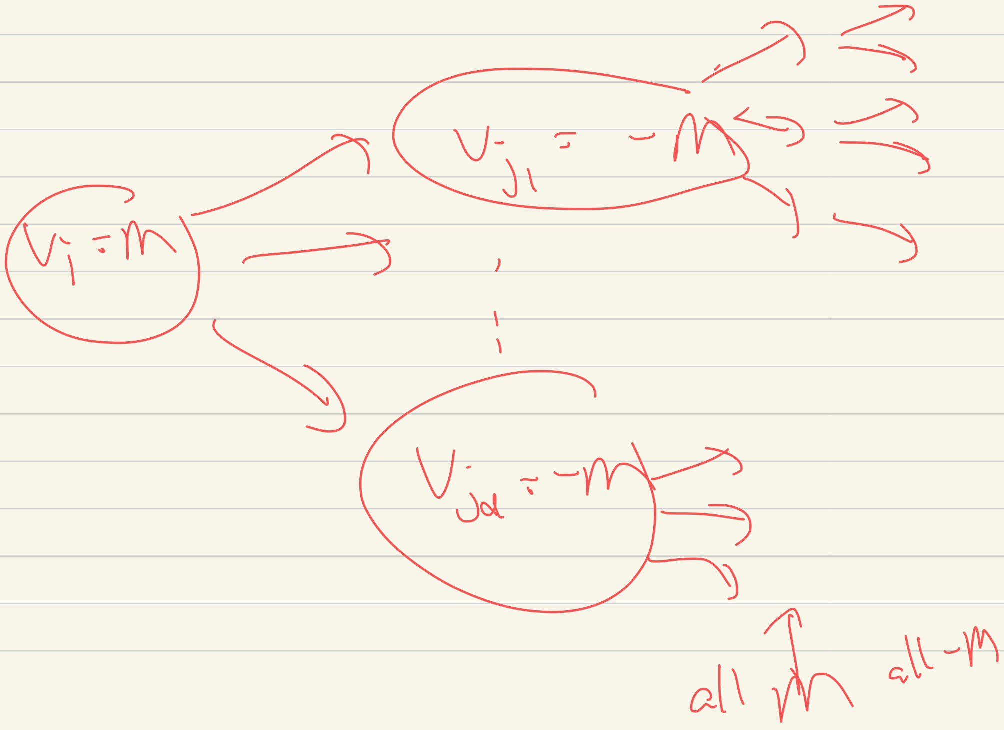


$\Rightarrow \quad v_s = m$ for all s connected to i.

$\Rightarrow$

$\lambda = -d,$

$$v_{j_1} = \cdots = v_{j_d} = -m$$

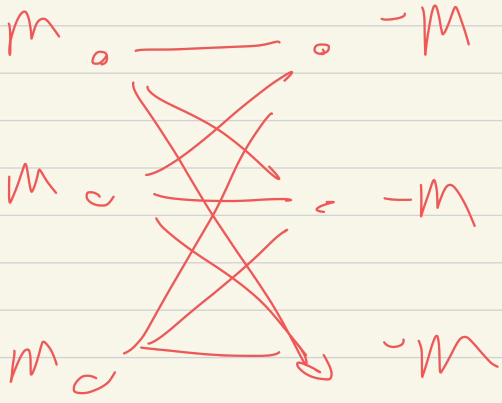


So then $v_j = \pm m$ for all

j connected to i,

and all paths from fixed j to

i are $\begin{cases} \text{all even} \\ \text{all odd} \end{cases}$



$\lambda = -3$

So if G d-regular,

$$-d \leq \lambda_n(G) \leq \dots \leq \lambda_2(G) \leq \lambda_1(G) = d$$

<u>Exercise</u>: # of connected comp of G = multiplicity of d as an eigenvalue

<u>Exercise</u>: # of connected comp of G that are bipartite = multiplicity of $-d$

Recall! A_G has an orthonormal set of eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$

$(A_G \vec{v}_i = \lambda_i \vec{v}_i)$ and

if $\vec{v} = c_1 \vec{v}_1 + \ldots + c_n \vec{v}_n$

then

$$A_G \vec{v} = c_1 \lambda_1 \vec{v}_1 + \ldots + c_n \lambda_n \vec{v}_n$$

and

$$A_G^k \vec{v} = c_1 \lambda_1^k \vec{v}_1 + \ldots + c_1 \lambda_n^k \vec{v}_n$$

so if $c_1 \neq 0$, and $\lambda_1 = d$ and

$$\rho = \max_{i \geq 2} |\lambda_i|$$

then

$$A_G^k \vec{v} = c_1 d^k \vec{v}_1$$

$$+ \left(\max_{i \geq 2} |c_i|\right)(n-1) \cdot \rho^k$$

$$= c_1 d^k \left(\vec{v}_1 + \text{error term}\right)$$

$$\|\text{error term}\| = \frac{\left\| \sum c_i \cancel{\lambda}_i^k \vec{v}_i \right\|}{|c_1 d^k|}$$

$$= O\left(\rho^k / d^k\right) \to 0$$

as $k \to \infty$.

Furthermore $\vec{v}_1, \dots, \vec{v}_n$

orthonormal, then

$$A = \sum_{i=1}^{n} \lambda_i \underbrace{\vec{v}_i \, \vec{v}_i^{\,T}}_{\text{orthog proj onto } \vec{v}_i}$$

e.g. $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$\begin{bmatrix} a & b \\ b & a \end{bmatrix}$; λ : $a+b$, $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$a-b$, $\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

then :

$$A = (a+b)\,\vec{v_1}\cdot\vec{v_1}^{T}$$
$$+(a-b)\,\vec{v_2}\cdot\vec{v_2}^{T}$$

$$= (a+b)\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$+(a-b)\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$= (a+b)\left(\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$$

$$+ (a-b)\left(\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)$$

Case $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $\quad a+b=3$

$a-b=1$

$$A = 3\left(\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$$

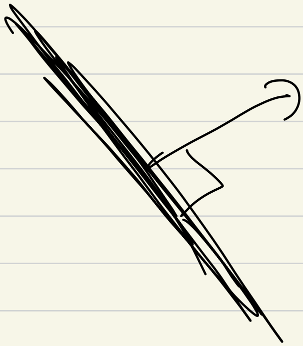
$$+ 1\left(\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)$$

$=$

Rem! $\displaystyle\sum_{i=1}^{n} \vec{v}_i \vec{v}_i^{\,T} = I$

(any ON basis $\vec{v}_1, \ldots, \vec{v}_n$)

Also $\underbrace{v_1 v_1^T}_{\text{Proj } \vec{v}_1} + \underbrace{\sum_{i \geq 2} v_2 v_2^T}_{\text{Proj } \vec{v}_1}$



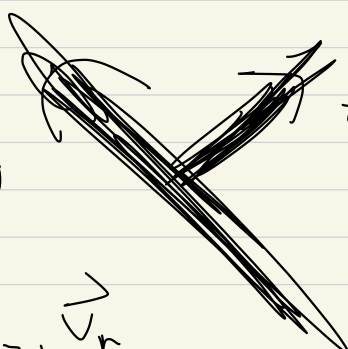
$\vec{v}_1$

← Orthog to $\vec{v}_1$

$$(\vec{v}_1)^{\perp}$$

Also

$$\sum_{i=1}^{n} \vec{v}_i \vec{v}_i^{T} = \sum_{i=1}^{k} \cdot + \sum_{i=k+1}^{n} \cdot$$



← span of $\vec{v}_1, \dots, \vec{v}_k$

Span of $\vec{v}_{k+1}, \dots, \vec{v}_n$

If G d-reg

$$\vec{v}_1 = \begin{bmatrix} 1/\sqrt{n} \\ \vdots \\ 1/\sqrt{n} \end{bmatrix},$$ then

$$\vec{v}_1 \vec{v}_1^T = \begin{bmatrix} 1/\sqrt{n} \\ \vdots \\ 1/\sqrt{n} \end{bmatrix} \left(1/\sqrt{n} \cdots 1/\sqrt{n}\right)$$

$$= \frac{1}{n} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$$ define

$$= \frac{1}{n} E_n$$

So $\lambda_1 = d \geq \lambda_2 \geq \cdots \geq \lambda_n$

then

$$\boxed{A_G} = \lambda_1 \vec{v}_1 \vec{v}_1^{\,T} + \sum_{i \geq 2} \lambda_i \vec{v}_i \vec{v}_i^{\,T}$$

$$\boxed{\frac{d}{n} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix} + \mathcal{E}}$$

$\mathcal{E}$ "error term"

$\mathcal{E}$ lives on $(\vec{\mathbb{1}})^{\perp}$,

and $\|\mathcal{E} v\|_{L^2} \leq \rho \|v\|_{L^2}$

$$\boxed{A_G^k = \underbrace{\frac{d^k}{n}\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix} + \mathcal{E}^k}_{\text{leads to "mixing lemma"}}}$$

where

$$\|\mathcal{E}^k\|_{L^2}$$

$$:= \max_{\vec{v}} \frac{\|\mathcal{E}^k \vec{v}\|}{\|\vec{v}\|}$$

$$= \rho^k$$

Note! $\overline{W}$ vector space, with

norm $\| \ \|$

e.g., $\mathbb{R}^n$, $\|v\|_{L^p} = \left(\sum_{i=1}^{n} |v_i|^p \right)^{1/p}$

so any $\| \ \|$ called a norm!

$\vec{w} \in \overline{W}$, returns real $\|\vec{w}\|$

s.t. (1) α scalar,

$\|\alpha \vec{w}\| = |\alpha| \, \|\vec{w}\|$

(2) $\|\vec{w}\| = 0 \iff \vec{w} = \vec{0}$

(3) $\|\vec{w}_1 + \vec{w}_2\| \leq \|\vec{w}_1\| + \|\vec{w}_2\|$

and $\mathcal{L} : W \to W$, we define

$$\|\mathcal{L}\| = \max_{\vec{w} \neq 0} \frac{\|\mathcal{L} w\|}{\|w\|}$$

then

$$\|\mathcal{L}^k\| \leq \|\mathcal{L}\|^k$$

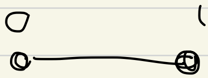
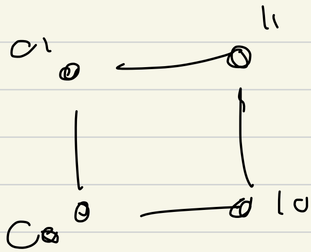
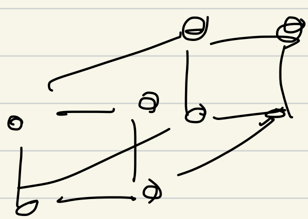
So $\|\mathcal{L}\| \leq \rho$

$$\Rightarrow \|\mathcal{L}^k\| \leq \rho^k$$

Break 10:21 – 10:26

$$\left[\text{"Discrete Fourier analysis" on abelian groups}\right]$$

Boolean Hypercube:

		V	E
$\mathbb{B}^1$		$\{0,1\}$	vertices differing by single coordinate
$\mathbb{B}^2$		$\{0,1\}^2$	
$\mathbb{B}^3$		$\{0,1\}^3$	

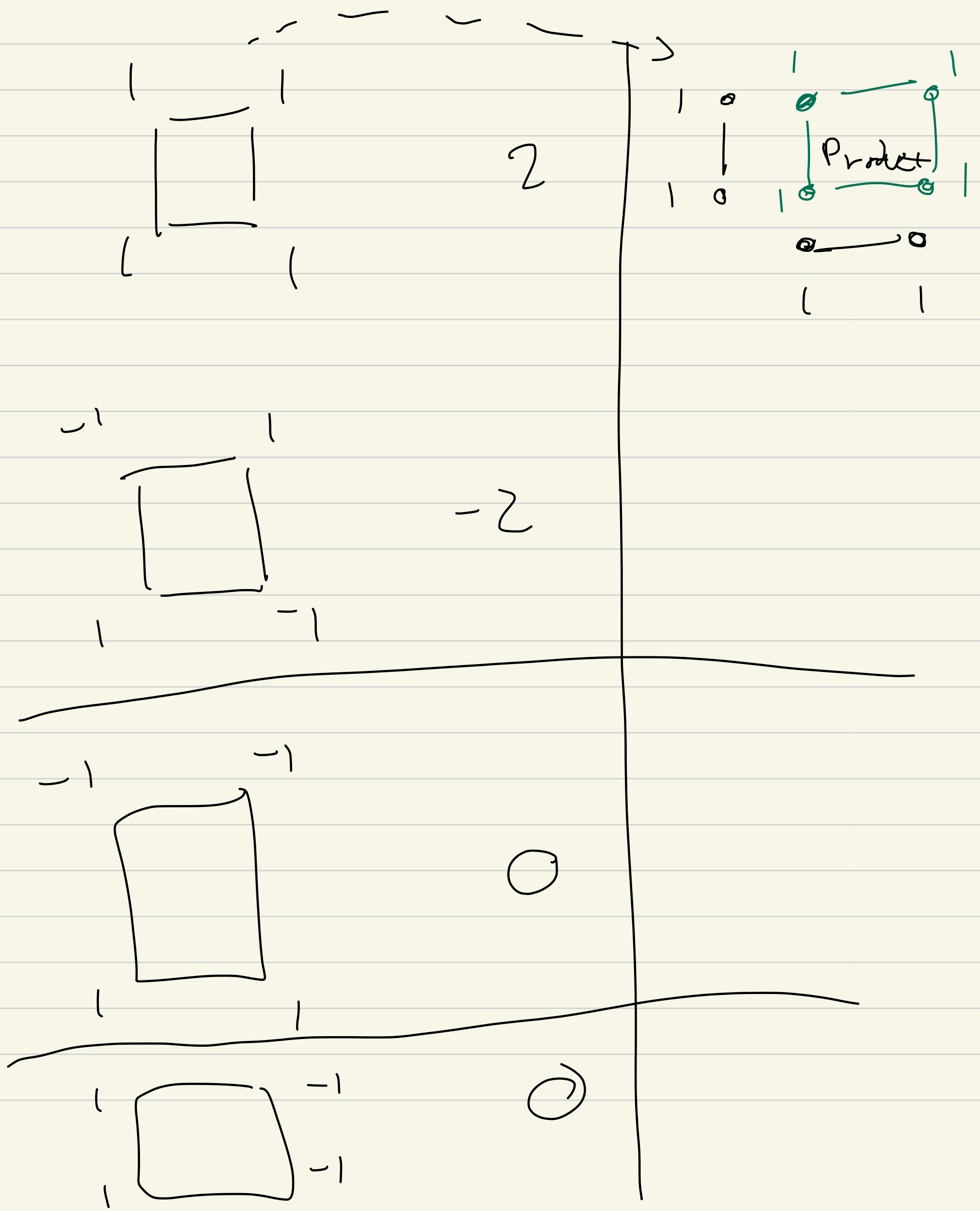
$$A_{B'} = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}$$

$$= \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \quad a = 0, \; b = 1$$

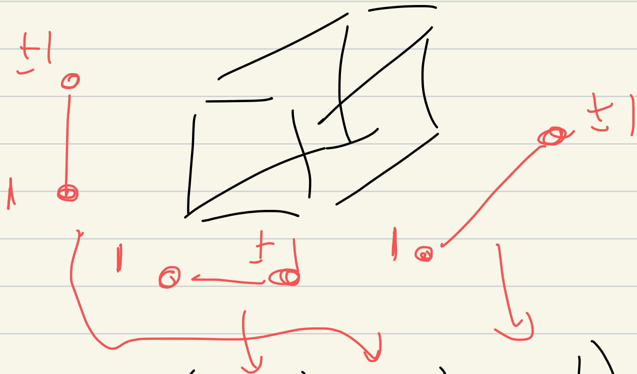
$$\lambda_1 = a + b = 1 \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix} / \sqrt{2}$$

$$\lambda_2 = a - b = -1 \qquad \begin{bmatrix} 1 \\ -1 \end{bmatrix} / \sqrt{2}$$

Eigenfunctions for $\mathbb{A}_{\mathbb{B}^2}$



For $A_{\mathbb{B}^3}$?



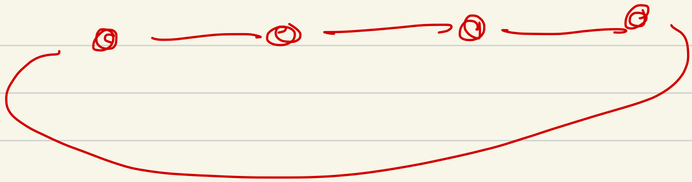
$$\mathbb{B}^3 = \text{Product}(\mathbb{B}^1, \mathbb{B}^1, \mathbb{B}^1)$$

eigenvalues/vectors given in terms of

$\mathbb{B}^1$

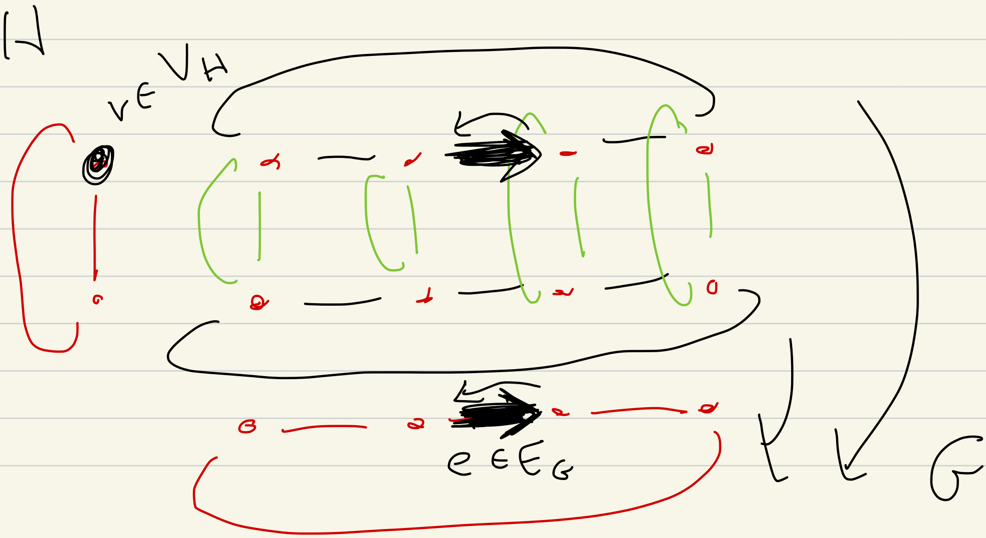
= Product (there are many products)

Also



2-regular , Cycle Length 4

Torus 2-dim



Def: If $G = (V_G, E_G, h_G, t_G, 2_G)$

$H = (V_H, E_H, h_H, t_H, 2_H)$

then cartesian product

$G \times H =$

vertizes: $V_G \times V_H$

edges: "horizontal" "vertical"

$$E_{G\times H} = \overbrace{E_G \times V_H} \amalg V_G \times E_H$$

where $(e,v) \in E_G \times V_H$ horizontal

$t_{G\times H}(e,v) = (te, v)$, $h_{G\times H}(e,v) = (he, v)$

$$\eta_{G \times H}(e, v) = (\eta_G e, v)$$

=

Claim: If $A_G \vec{x} = \lambda \vec{x}$,

for $\vec{x} : V_G \to \mathbb{R}$,

$A_H \vec{y} = \mu \vec{y}$, $\vec{y} : V_H \to \mathbb{R}$

then

$(\vec{x} \otimes \vec{y})$ defined as

$$(\vec{x} \otimes \vec{y})(v_1, v_2) = x(v_1)\, y(v_2)$$

in V_G $\quad$ V_H

Then

$$A_{G\times H}(\vec{x}\otimes\vec{y})$$

$$=(\lambda+\mu)(\vec{x}\otimes\vec{y})$$

$=$

$$\begin{array}{llll} & & & \lambda \\ \text{So} \quad \mathbb{B}^1 \ ! & A_{\mathbb{B}^1} & \begin{pmatrix}1\\1\end{pmatrix} & 1 \\ & & \begin{pmatrix}1\\-1\end{pmatrix} & -1 \end{array}$$

then

$$A_{\mathbb{B}^3}\left\{\begin{array}{l} \begin{pmatrix}1\\1\end{pmatrix}\otimes\begin{pmatrix}1\\1\end{pmatrix}\otimes\begin{pmatrix}1\\1\end{pmatrix} \\ \begin{pmatrix}1\\1\end{pmatrix}\otimes\begin{pmatrix}1\\1\end{pmatrix}\otimes\begin{pmatrix}1\\-1\end{pmatrix} \\ \vdots \\ \begin{pmatrix}1\\-1\end{pmatrix}\otimes\begin{pmatrix}1\\-1\end{pmatrix}\otimes\begin{pmatrix}1\\-1\end{pmatrix} \end{array}\right. \qquad \begin{array}{c}\lambda\\ \text{sum}\end{array}$$