CPSC 536F   Feb 17

Background on linear algebra, and

Expander Graphs and their Applications

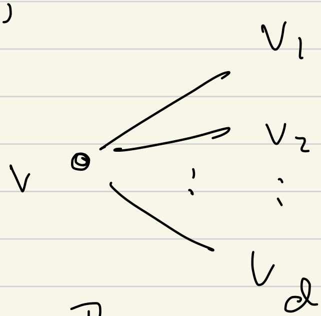
Hoory, Linial, Wigderson

See CPSC 531F References page (I'll put a link)

Start expanders.

Expanders: think of a graph, $G=(V,E)$

$|V|=n$,



"d-regular"

Meaning each v is incident upon d edges.

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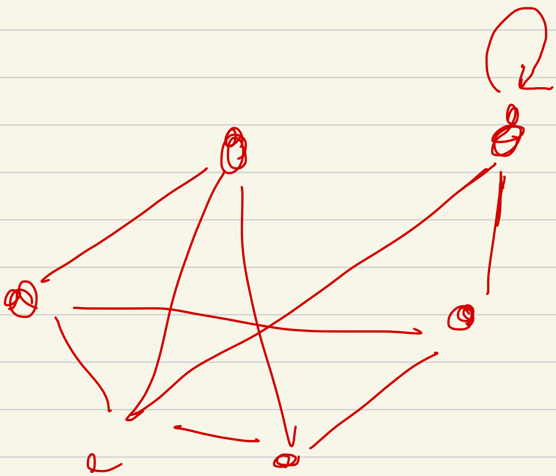
Think n large, d is fixed (or slowly growing with n).

Application!

??

Network of Computers



3 regular

=

Graphs!

Directed graph!

$$G = (V, E, h, t)$$

$$= (V_G, E_G, h_G, t_G)$$

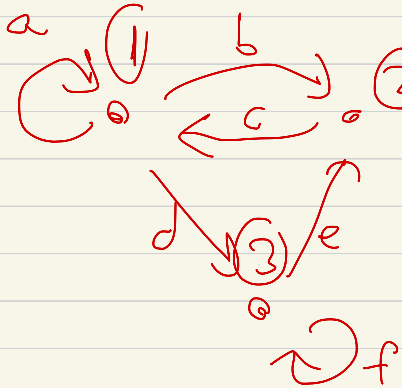
$G = (V, E)$ sets

h "heads map" $E \to V$

t "tails map" $E \to V$

e.g.,

$V = \{1, 2, 3\}$



$E = \{a, b, c, d, e, f\}$

$$h: \begin{array}{lll} a & \longmapsto & 1 \\ b & \longmapsto & 2 \\ c & & \\ d & & \text{etc.} \\ e & & \\ f & & \end{array}$$

$$f: \begin{array}{lll} a & \longmapsto & 1 \\ b & \longmapsto & 1 \\ c & \longmapsto & 2 \\ & \vdots & \end{array}$$

Each digraph has an adjacency matrix:

$$A_G = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

a_{ij} (often) # edges from $i \to j$

$$= \left| \left\{ e \in E \;\middle|\; \begin{array}{l} h(e) = j \\ t(e) = i \end{array} \right\} \right|$$

If A acts on columns,

better

$$A_{ij} = \#\text{ edges } j \to i$$

(For graphs this doesn't matter)

$$A_G = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

other convention

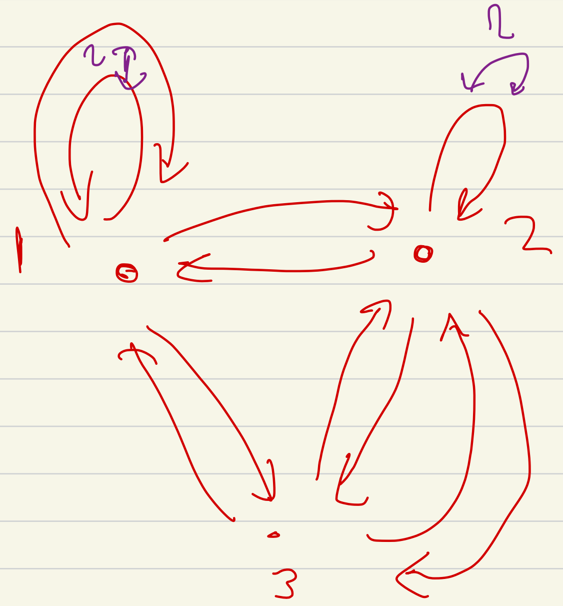
$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

A <u>graph</u> is

$$G = (\underbrace{V, E, h, t}_{\text{digraph}}, \iota)$$

ι: pairing of edges

ι



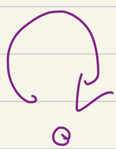
s.t. V, E, h, t is a digraph, and $\tau: E \to E$ "pairing" $\tau\tau = \mathrm{Id}$ and

$$t(\tau e) = he$$

τ "pairs the edges" and reverses orientation.

This gives you the usual notion of a multigraph (multiple edges, self-loops allowed) but

self-loop $= \{ e \in E \mid he = te \}$

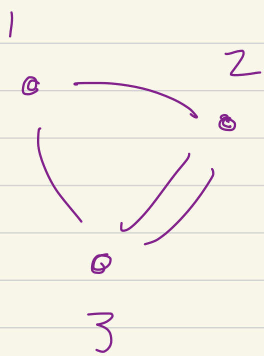


$\imath$!

$\imath(e_1) = e_1$, $\imath(e_2) = e_3$, $\imath(e_3) = e_2$

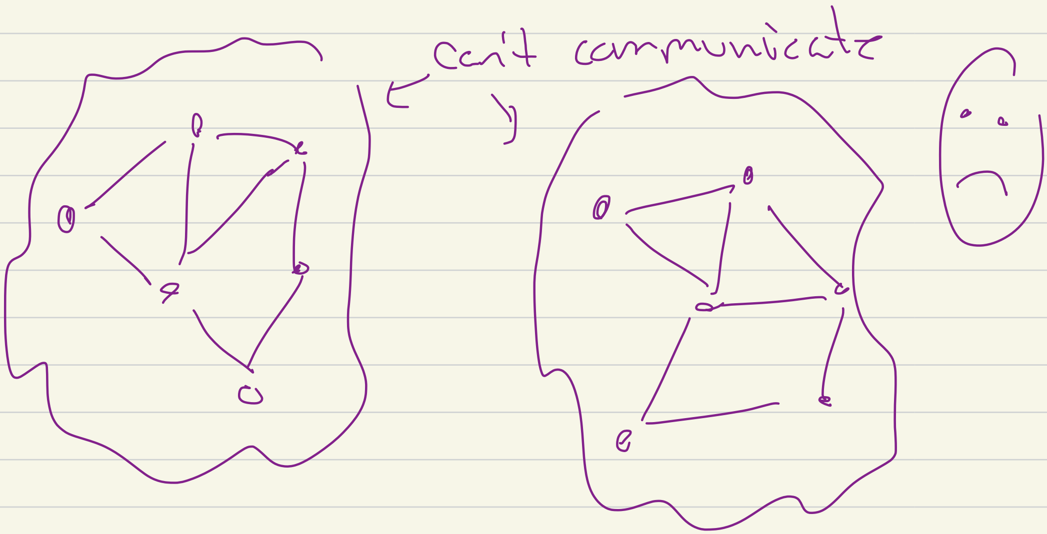
One also writes:

$$\iota(e_2) = e_3 \; : \qquad \underbrace{e_2^{-1} = e_3}_{\text{notation}}$$



$$A = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & & \left[\begin{array}{ccc} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{array}\right] & \\ 3 & & & \end{array}$$

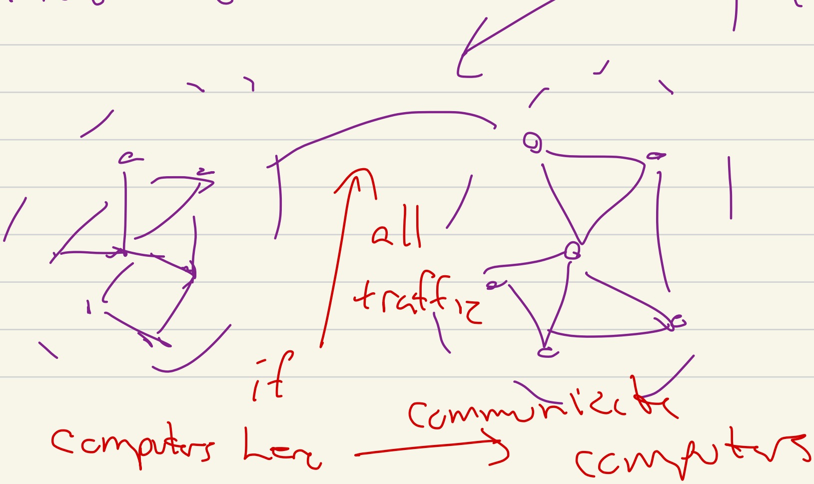
Telephone/Computer Network



2 connected components

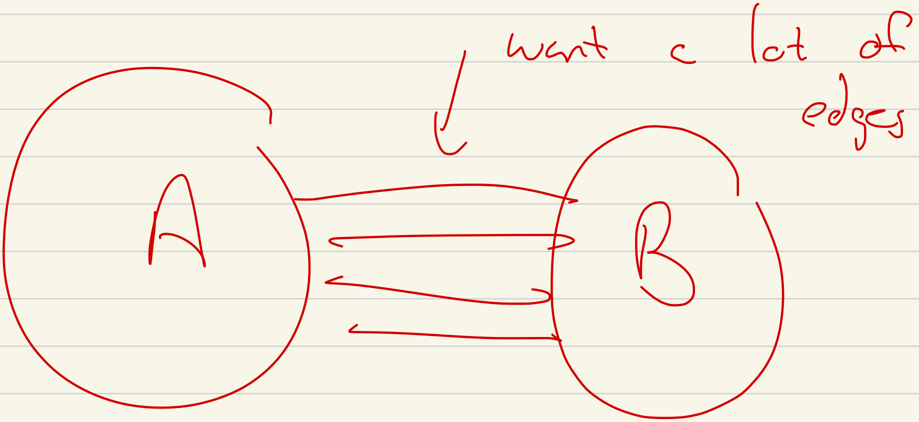
Almost as bad!

only 1 edge



Good interconnectivity means

V partitioned subsets A, B



Expander: Fix d, n large

s.t. say $|A| = \frac{n}{2}$ $|B| = \frac{n}{2}$

then

$\#$ edges A to B

$$= \left\{ e \in E \;\middle|\; \begin{array}{l} \cancel{t}(e) = A \\ h(e) = B \end{array} \right\}$$

is large (as large as possible) for any A, B, $A, B \subset V$, $|A| = |B| = \frac{n}{2}$.

Claim:

Say that G is a d-regular graph on n ~~v~~ vertices

d-edges

incident

upon

each

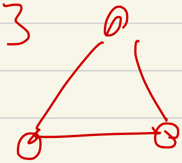
v.

$$\operatorname{degree}_G(v) = \#\{e \in E \mid he = v\}$$

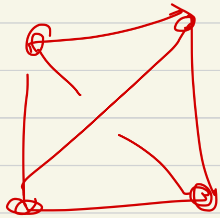
$$= \#\{e \in E \mid te = v\}$$

Example:

K_3



K_4

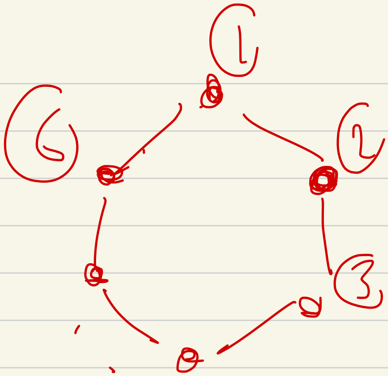


Complete graph K_n on

n-vertices, no self loops

$$A_{K_3} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \leftarrow \text{2-regular}$$

$$A_{K_4} = \begin{pmatrix} 0 & & 1\text{'s} & \\ & 0 & & \\ 1\text{'s} & & 0 & \\ & & & 0 \end{pmatrix} \leftarrow \text{3-regular}$$



← cycle of size n

C_6

$$
\begin{array}{c}
\begin{array}{cccccc} 1 & & - & - & & 6 \end{array} \\
A_G = \begin{bmatrix}
0 & 1 & & & & 1 \\
1 & 0 & 1 & & & \\
& 1 & 0 & 1 & & \\
& & \ddots & \ddots & \ddots & \\
& & & 1 & \ddots & 1 \\
1 & & & & 1 & 0
\end{bmatrix}
\end{array}
$$

= 2-regular

Claim: Eigenvalues of A_G allow us to measure "interconnectivity"

A_G adjacency matrix of G

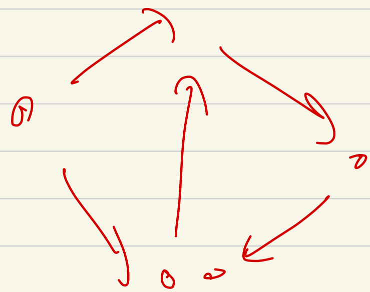
$(A_G)_{ij}$ = # edges from $i \to j$

$(A_G^2)_{ij}$ = # walks of length 2 $i \to j$

$\vdots$

$(A_G^k)_{ij}$ = # walks of length k from $i \to j$

A walk in a digraph



from v to v'

length k

is $e_1, \ldots, e_k$

s.t.

$\bullet \longrightarrow \longrightarrow \cdots \longrightarrow \bullet$

$v = t(e_1)$, $h(e_k) = v'$

$h(e_1) = t(e_2)$ $h(e_2) = t(e_3)$ ...

$h(e_1) = t(e_2)$

$v \; \bullet \xrightarrow{e_1} \; \xrightarrow{e_2} \; \longrightarrow \cdots \longrightarrow \bullet \; v'$

A_G = symmetric matrix

G = graph

G d-regular $\iff$

A_G $\leftarrow$ all rows sums

all column sums

= d

$\iff$

$$A_G \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = d \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$A_G \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \text{first row } A_G \cdot (1, \dots, 1) \\ \text{2nd row } A_G \quad " \quad " \\ \\ n^{th} \text{ row } A_G \quad " \quad " \end{bmatrix}$$

$$= \begin{pmatrix} \deg_G(v_1) \\ \deg_G(v_2) \\ \vdots \\ \deg_G(v_n) \end{pmatrix}$$

$$A_G \text{ d-regular} \quad = \begin{pmatrix} d \\ \vdots \\ \vdots \\ d \end{pmatrix}$$

Claim: G is d-regular, then

$$A_G = \frac{d}{n}\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \\ 1 & & 1 \end{bmatrix} + \mathcal{E}$$

where $\mathcal{E}$ "lives on $\vec{\mathbb{1}}^{\perp}$"

and $\|\mathcal{E}\|_{L^2}$ = second largest eigenvalue in absolute value

expander (from eigenvalue def)

Recall: If A is any $n \times n$ symmetric matrix, then there exist $\vec{v}_1, \ldots, \vec{v}_n$ s.t.

1. $\vec{v}_i \cdot \vec{v}_j = 0$

2. $A\vec{v}_i = \lambda_i \vec{v}_i$

for some real λ_i,

$\lambda_1, \ldots, \lambda_n$ = eigenvalues

Remark: $A\vec{v} = \lambda \vec{v}$

then $= \lambda I \vec{v}$

so

$\det(A - \lambda I) = 0$

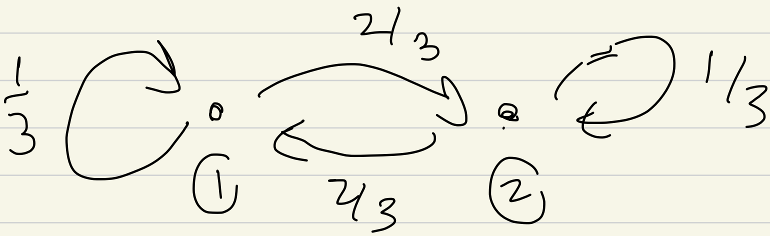
this "theoretically" determines all eigenvalues.

Typically you need other methods to find $\lambda_1, \ldots, \lambda_n$ or the "most significant ones"

2×2:

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

think of



here think of $a, b \geq 0$,

often $a + b = 1$ for a Markov

Matrix.

Then

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ a+b \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

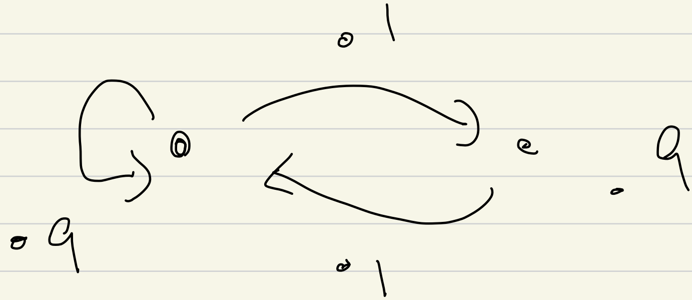
$$\lambda = a+b$$

Then

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a-b \\ b-a \end{bmatrix}$$

$$= \lambda' \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda' = a-b$$



$$P = \begin{bmatrix} .9 & .1 \\ .1 & .9 \end{bmatrix}$$

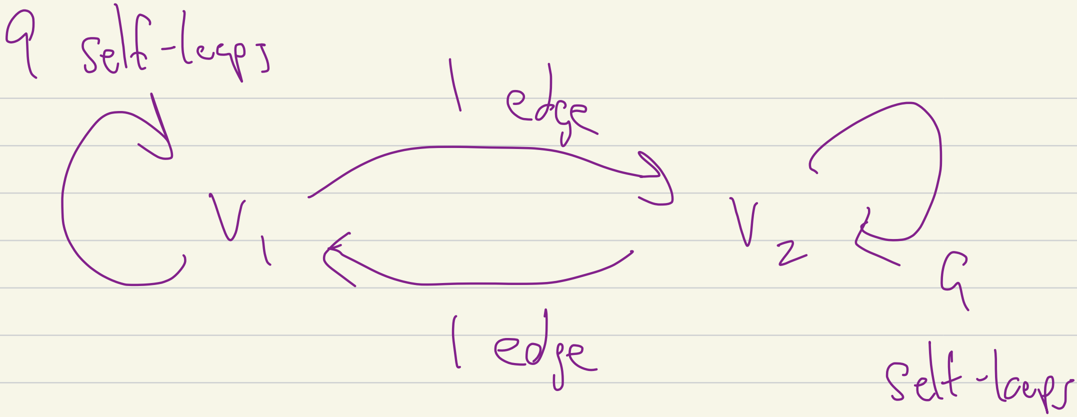
λ's! $1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\underbrace{(.9 - .1)}_{.8} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Thm!

$$\mathrm{Trace}(A^k) = (A^k)_{1,1} + \cdots + (A^k)_{n,n}$$

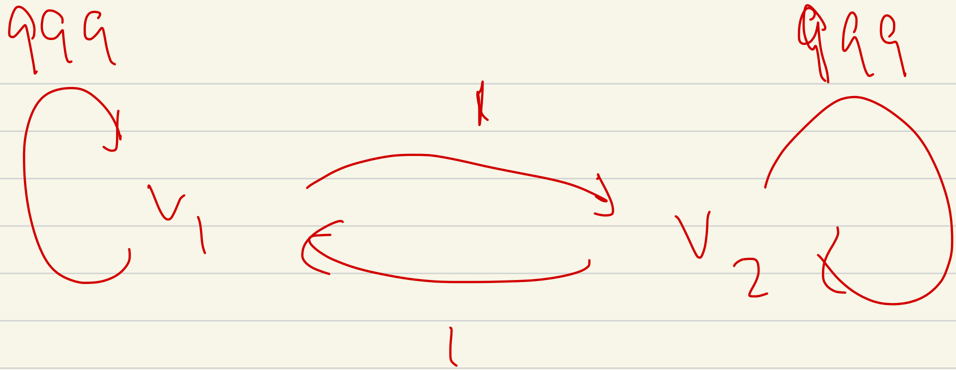
$$= \lambda_1 + \lambda_2 + \_\_ + \lambda_n$$



$$A_G = \begin{bmatrix} 9 & 1 \\ 1 & 9 \end{bmatrix} = 10P,$$

$$\lambda = 10, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ G is 10-regular

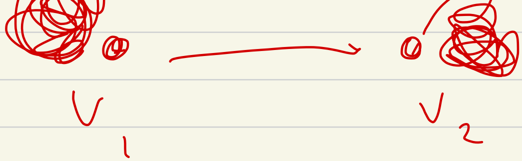
$$\lambda = 8, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

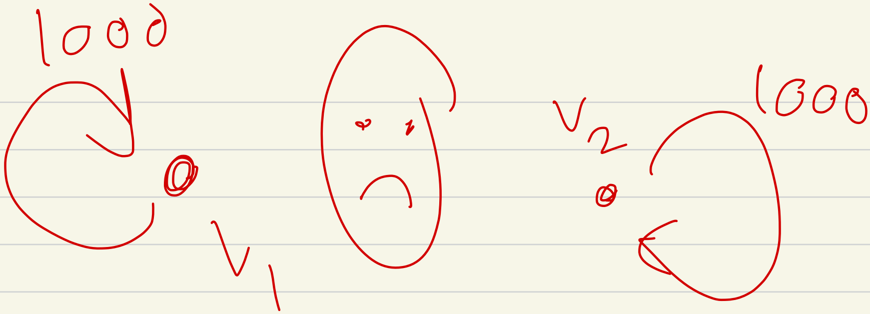


$$\lambda_1 = 1000 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 998 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

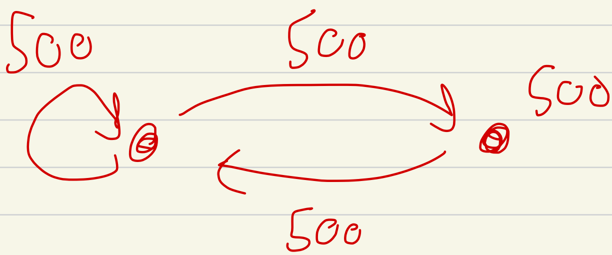
"bad expander"





$$A_G = \begin{bmatrix} 1000 & \\ & 1000 \end{bmatrix}$$

$$\lambda\text{'s} = 1000,\ 1000$$



$$1000, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$500 - 500 = 0, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A_G = \begin{bmatrix} 500 & 500 \\ 500 & 500 \end{bmatrix}$$

Break 10:28 – 10:33

Thm: If A is $n \times n$ symmetric ($A^T = A$ or $a_{ij} = a_{ji}$) then there exists an orthonormal basis $\vec{v}_1, \ldots, \vec{v}_n$ s.t. $A\vec{v}_i = \lambda_i \vec{v}_i$ for some real $\lambda_1, \ldots, \lambda_n$.

Orthonormal:

$$\vec{v}_i \cdot \vec{v}_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

In this case:

$$\vec{v} = c_1 \vec{v}_1 + \ldots + c_n \vec{v}_n$$

$$\vec{v}_j \cdot \vec{v} = (\qquad) \cdot \vec{v}_j$$

$$= c_1 (\vec{v}_1 \cdot \vec{v}_j) + \ldots \qquad + c_n (\vec{v}_n \cdot \vec{v}_j)$$

$$= \qquad c_j$$

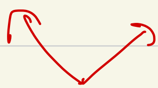
So: $c_j = \vec{v} \cdot \vec{v}_j$

So

$$\vec{v} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n$$

$$= (\vec{v}_1 \cdot \vec{v})\vec{v}_1 + \dots + (\vec{v}_n \cdot \vec{v})\vec{v}_n$$

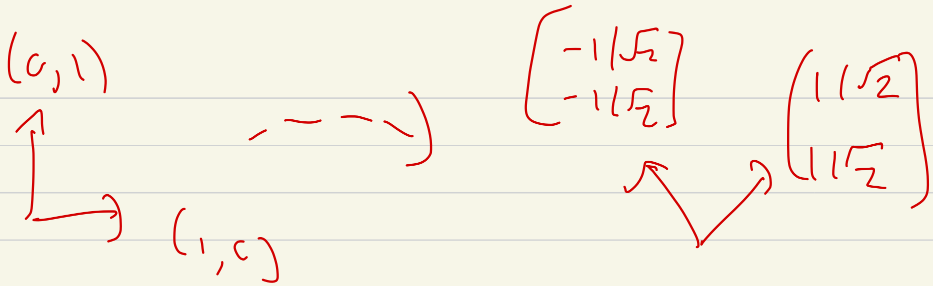
Think of $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ normalized

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Big/ \left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\|$$

$$\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \qquad = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$



$$A^k \vec{v}$$

$$= A^k \left(c_1 \vec{v}_1 + \dots + c_n \vec{v}_n \right)$$

$$= c_1 A^k \vec{v}_1 + \dots + c_n A^k \vec{v}_n$$

$$= c_1 \lambda_1^k \vec{v}_1 + \dots + c_n \lambda_n^k \vec{v}_n$$

If A is $n \times n$ symmetric, real

then

$$R_A(\vec{v}) = \frac{(A\vec{v}) \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$$

Rayleigh quotient.

Assume the spectral thm.

$$\vec{v} = c_1 \vec{v}_1 + \cdots + c_n v_n$$

$$R_A(\vec{v}) = \frac{c_1^2 \lambda_1 + \cdots + c_n^2 \lambda_n}{c_1^2 + \cdots + c_n^2}$$

$$\vec{v} \cdot \vec{v} = (c_1 \vec{v}_1 + \dots + c_n \vec{v}_n) \cdot (c_1 \vec{v}_1 + \dots + c_n \vec{v}_n)$$

$$= c_1^2 + c_2^2 + \dots + c_n^2$$

$$(A\vec{v}) \cdot \vec{v}$$

$$= (\lambda_1 c_1 \vec{v}_1 + \dots + \lambda_n c_n \vec{v}_n) (c_1 \vec{v}_1 + \dots + c_n \vec{v}_n)$$

$$= c_1^2 \lambda_1 + \dots + c_n^2 \lambda_n$$

So:

largest

$$\lambda_n \leq \dots \leq \lambda_2 \leq \lambda_1$$

smallest

$$R_A(\vec{v}) = \frac{c_1^2 \lambda_1 + \dots + c_n^2 \lambda_n}{c_1^2 + \dots + c_n^2}$$

$$\leq \frac{c_1^2 \lambda_1 + \dots + c_n^2 \lambda_1}{c_1^2 + \dots + c_n^2} \leq \lambda_1$$

Thm: Largest possible value of $R(v) = \lambda_1$

Eigenvalues:

$$K_4 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

trick:

$$K_4 + I = \begin{pmatrix} 1 & - & - & 1 \\ \vdots & & & \\ 1 & - & - & 1 \end{pmatrix} = E_4$$

If $\vec{v} \perp \vec{1}$ i.e. $\vec{v} \cdot \vec{1} = 0$

$$v_1 + v_2 + \cdots + v_n = 0$$

Then

$$\underbrace{\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}}_{E_n} \vec{v} = \begin{bmatrix} v_1 + \cdots + v_n \\ \vdots \\ v_1 + \cdots + v_n \end{bmatrix}$$

$$= \vec{0}$$

$$\underbrace{\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}}_{E_n} \underbrace{\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}}_{\vec{1}} = n \cdot \vec{1}$$

Hence:

$$E_n \quad \begin{pmatrix} 1 & \cdots & 1 \\ & \ddots & \\ 1 & & 1 \end{pmatrix}$$

has λ's $\quad n \quad$ mult. 1

$0 \quad$ mult. $n-1$

$$\underbrace{\lambda_n = \cdots = \lambda_2}_{0} \leq \underset{n}{\overset{\shortparallel}{\lambda_1}}$$

$$(A+I)\vec{v} = \lambda \vec{v}$$

$$A\vec{v} + \vec{v} = \lambda \vec{v}$$

$$A\vec{v} = (\lambda - 1)\vec{v}$$

Then

$$(K_n + I) = \begin{pmatrix} 1 & \cdots & 1 \\ & \ddots & \\ 1 & & 1 \end{pmatrix}$$

λ	mult
n	1
0	$n-1$

$K_n =$

λ	mult
$n-1$	1
-1	$n-1$

$$\underbrace{\lambda_n = \cdots = \lambda_2}_{-1} \qquad \lambda_1(K_n) = n-1$$