CPSC 536F Feb 3, 2022

Class on Feb 8 begins in-person, ICCS Room 246

Today:

(1) $\text{Threshold}_k(x_1, \ldots, x_n)$ is monotone $O(n \log n)$

(2) (Valiant) monotone $\widehat{\text{Maj}}(x_1, \ldots, x_n)$ in poly(n) (we'll do $n^{6.29\ldots+\varepsilon}$, Valiant does roughly $n^{5.3\ldots}$)

Start:

Valiant's paper on counting perfect matchings (0/1 permanent) is #P complete:

"The Complexity of Computing the Permanent," L.G. Valiant, Theoretical Computer Science, 1979

Boolean functions
Monotone Boolean functions
Algebraic functions

} Spira's lemma holds

Boolean function $f(x_1, \ldots, x_n)$ depends on all the x_i's!

min formula size $\geq n$

" " depth $\geq \log n$

Best depth bound to date on a function in NP is $\geq c \cdot \log n$

For formula depth $\cong$ circuit depth

Typical formula requires $\geq 2^n / \log n$ size, even a balanced full tree has depth $\geq \log_2(2^n / \log n) = n - \log\log n$

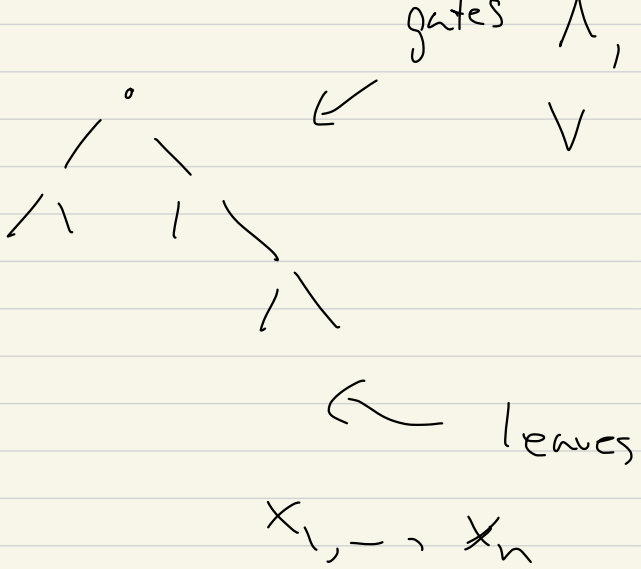
Result on monotone circuits is

there are superpolynomial lower

bounds on circuit size

Razborov, Alon-Boppana

Monotone formula



Monotone function:

$$f : \{0,1\}^n \to \{0,1\}$$

s.t.

if $f(\vec{x}) = 1$, $\vec{x} \leq \vec{y}$

(i.e. $x_i \leq y_i$ for all i)

$\Rightarrow$ $f(\vec{y}) = 1$.

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n even

$\vec{x}$ $\quad > \frac{n}{2} \Big\{ f = 1$

$x_1 + x_2 + \cdots + x_n = \frac{n}{2}$ | arbitrary 0/1

$\quad\quad < \frac{n}{2} \Big\} f = 0$

So # monotone functions

$x_1, \ldots, x_n$

$$\geq \binom{n}{n/2}$$

$$\geq 2^{\left(2^n/\sqrt{n}\, c\right)}$$

close to

$$2^{\left(2^n\right)}$$

Probabilistic method:

$$Th_k(x_1,\dots x_n) = \begin{cases} 1 & \text{if } x_1+\dots+x_n \geq k \\ 0 & \text{if } \dots < k \end{cases}$$

$Th_2(x_1,\dots,x_n)$ has $n \log_2 n$

formula:

- Kyle divide and conquer

- $Th_2 = \bigvee_{i<j} x_i \text{ AND } x_j$

size $\binom{n}{2} \cdot 2$

$(X_1 \text{ OR } X_2 \text{ OR } X_3)$ AND $(X_4 \text{ OR } X_5)$

$= X_1 \wedge X_4$ OR $X_1 \wedge X_5$ OR

$X_2 \wedge X_4 \vee X_2 \wedge X_5 \vee$

$X_3 \wedge X_4 \vee X_3 \wedge X_5$

"h_2: want $i \neq j$, then $\{1, \ldots, n\}$

(OR OR) AND (OR OR) OR

(X_i) AND (X_j) :

( ) ( ) :

Th_2:

$x_0 \leftrightarrow$ 000 $\leftarrow 0$

$\vdots$ 001 $\leftarrow 1$

$x_7 \leftrightarrow$ 010 $\leftarrow 2$

011

$\vdots$

111

$Th_2(x_0, \ldots, x_7)$

$=$

(x_i s.t. (1st bit $i = 0$)) AND (x_j s.t. (1st bit $j \neq 1$))

$\vdots$

$=$

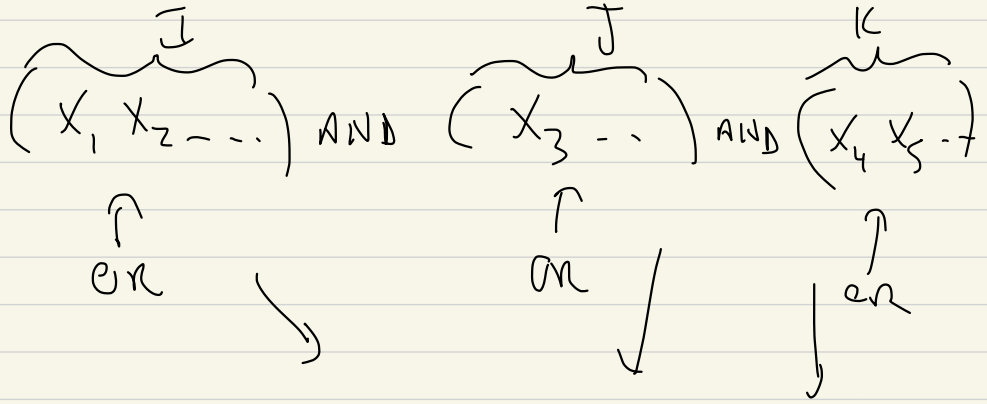
$Th_3(x_1, \ldots, x_n)$ recursion $O(n \log^2 n)$

Th_4 - - - $\vdots$ $O(n \log^3 n)$

$\vdots$

$Th_3(x_1, \ldots, x_n)$:

divide $\{1, \ldots, n\}$ into 3 groups



$$= \bigvee_{\substack{i \in I \\ j \in J \\ k \in K}} x_i \text{ AND } x_j \text{ AND } x_k$$

Idea: Take OR (I_1, J_1, K_1)
OR (I_2, J_2, K_2)
$\vdots$
(I_r, J_r, K_r)

The OR of

clause $(\bar{I}_1, \bar{J}_1, K_1)$

OR

clause $(\bar{I}_2, \bar{J}_2, K_2)$

OR

⋮

OR

clause $(\bar{I}_r, \bar{J}_r, K_r)$

gives

OR X_i AND X_j AND X_k

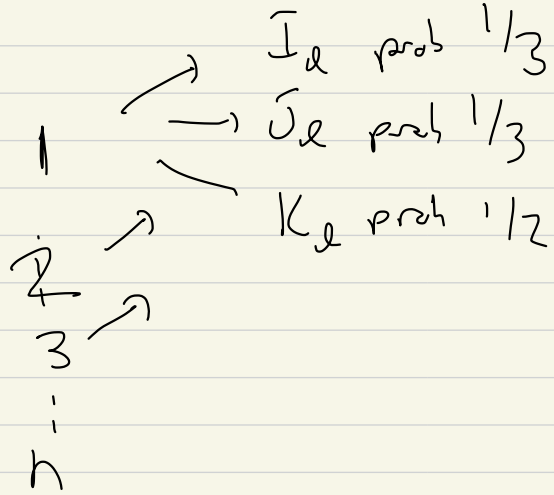
s.t. for some ℓ, $i \in \bar{I}_\ell, j \in \bar{J}_\ell, k \in K_\ell$

Probabilistic method we'll prove the existence of such

$(I_1, J_1, K_1), \dots (I_r, J_r, K_r)$:

a, b, c distinct in $\{1, \dots, n\}$

I_ℓ, J_ℓ, K_ℓ



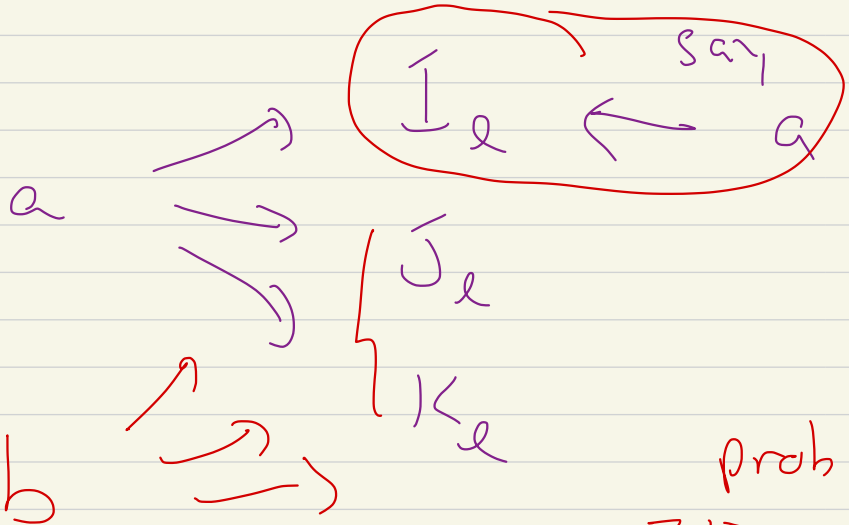
independently

we could put $1, \dots, n$ into I_ℓ with prob $\left(\frac{1}{3}\right)^n$

Claim:

a, b, c seperated by I_ℓ, J_ℓ, K_ℓ

with prob $2/q$



prob $1/3$ goes into ~~most~~ partition

prob $2/3$, b is in different partition

$$\left(\begin{array}{c}\text{Prob that } a,b,c \\ \text{not separated}\end{array}\right) = \frac{7}{9}$$

So $(I_1, J_1, K_1) - - (I_r, J_r, K_r)$

r times same experiment, independently

Or fraction of all $(3^n)^r$ possible partitions

$$\left(\begin{array}{c}\text{Prob that } a,b,c \\ \text{not separated} \\ \text{by } r \text{ partitions}\end{array}\right) = \left(\frac{7}{9}\right)^r$$

Union bound:

If $E_{a,b,c}$ = event that a, b, c is not separated

$$\text{Prob}\left[\bigcup_{a,b,c} E_{a,b,c}\right] \leq \sum_{a<b<c} \text{Prob}(E_{a,b,c})$$

$$\leq \binom{n}{3}\left(\frac{7}{9}\right)^r$$

Counting "bad partitions": 3^n possible I_1, J_1, K_1, 3^n poss I_2, J_2, K_2, ...

By comparison:

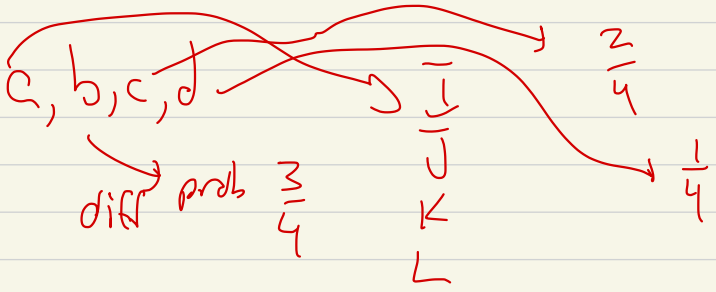
Th_2 : $a, b \in \{1, \ldots, n\}$

Prob bad event $I \cup J$

$\leq \binom{n}{2} \left(\frac{1}{2}\right)^r$

Th_4 not sep $1 - \frac{3}{4} \cdot \frac{2}{4} \cdot \frac{1}{4}$

$\leq \binom{n}{4} \left(\quad \right)^r$



$\Rightarrow$ Th_3 : formula doesn't work with prob

$$\leq \binom{n}{3}\left(\frac{7}{9}\right)^r$$

then

$$\frac{n(n-1)(n-2)}{6} < n^3\left(\frac{7}{9}\right)^r$$

if r chosen s.t. $n^3\left(\frac{7}{9}\right)^r \leq 1$

Then prob failure < 1

$$\left(\frac{q}{7}\right)^r > n^3$$

$$r \geq 3 \frac{\log_2 n}{\log_2 q/7}$$

=

Th$_k$ similarly

=

Th$_3$ take r s.t. $\left(\frac{7}{q}\right)^r \leq \frac{1}{n^4}$

$$r \geq 4 \frac{\log_2 n}{\log_2 q/7}$$

= but prob failure $\leq n^3/n^4 = \frac{1}{n}$

= How to find one explizitly is hard...

Similarly

almost all Boolean functions

$f : \{0,1\}^n \to \{0,1\}$

require size formula.

$$\left(2^n / \log n\right)\left(\frac{1}{n}\right)$$

add $\int$

Finding an explicit formula $\int$

(~~or NP~~)

requires this size

Valiant: Monotone formula

for $MAJ(x_1, \ldots, x_n)$ of

poly size.

Idea: say

n odd, $MAJ = 1$ if $\#x_1, \ldots, x_n$

$= 1$ is $\geq \frac{n+1}{2}$

0 if $\leq \frac{n-1}{2}$

Imagine: you pick 3 $x_1, \ldots, x_n$

at random, y_1, y_2, y_3, $MAJ(y_1, y_2, y_3)$

Then you get a better prob

than $\frac{(n+1)/2}{n} = \frac{1}{2} + \frac{1}{2n}$

of getting a 1 if majority

$x_1, \ldots, x_n$ are $= 1$

Then $z_1 = MAJ(y_1, y_2, y_3)$

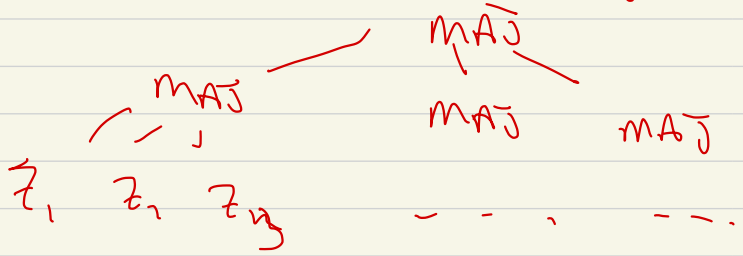
$z_2 = MAJ(y_4, y_5, y_6)$

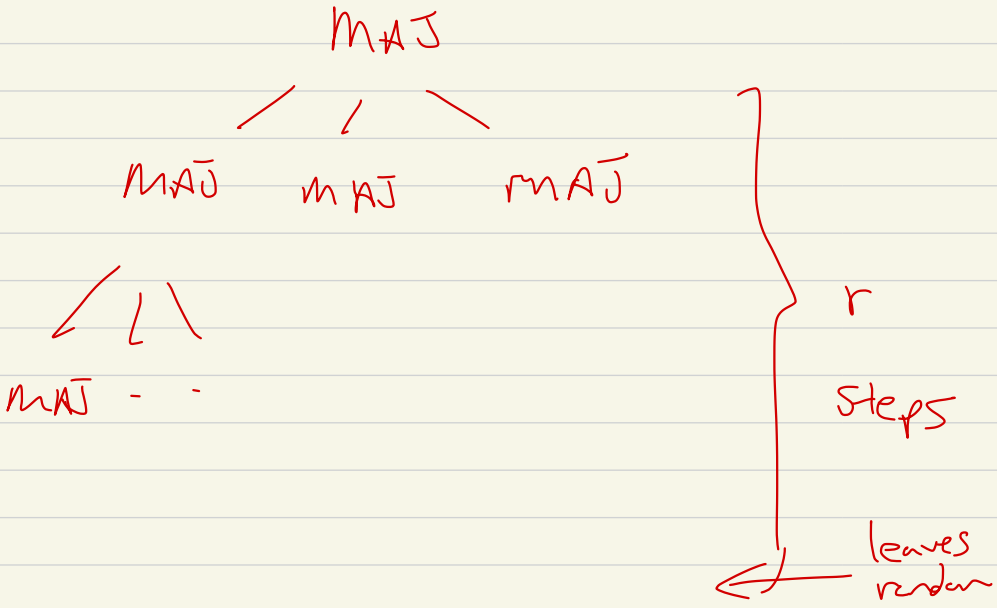
$z_3 = MAJ(y_7, y_8, y_9)$

$MAJ(z_1, z_2, z_3)$

continue

Say we have iterated majority:





Then: If $\Pr[X_1, \ldots, X_n = 0] \geq \frac{1}{2} + \frac{1}{2n}$

then r steps it MAJ is 0 with prob.

$\geq 1 - \frac{1}{2^n}$

pairs $x_1, \ldots, x_n = 2^n$

Want to show:

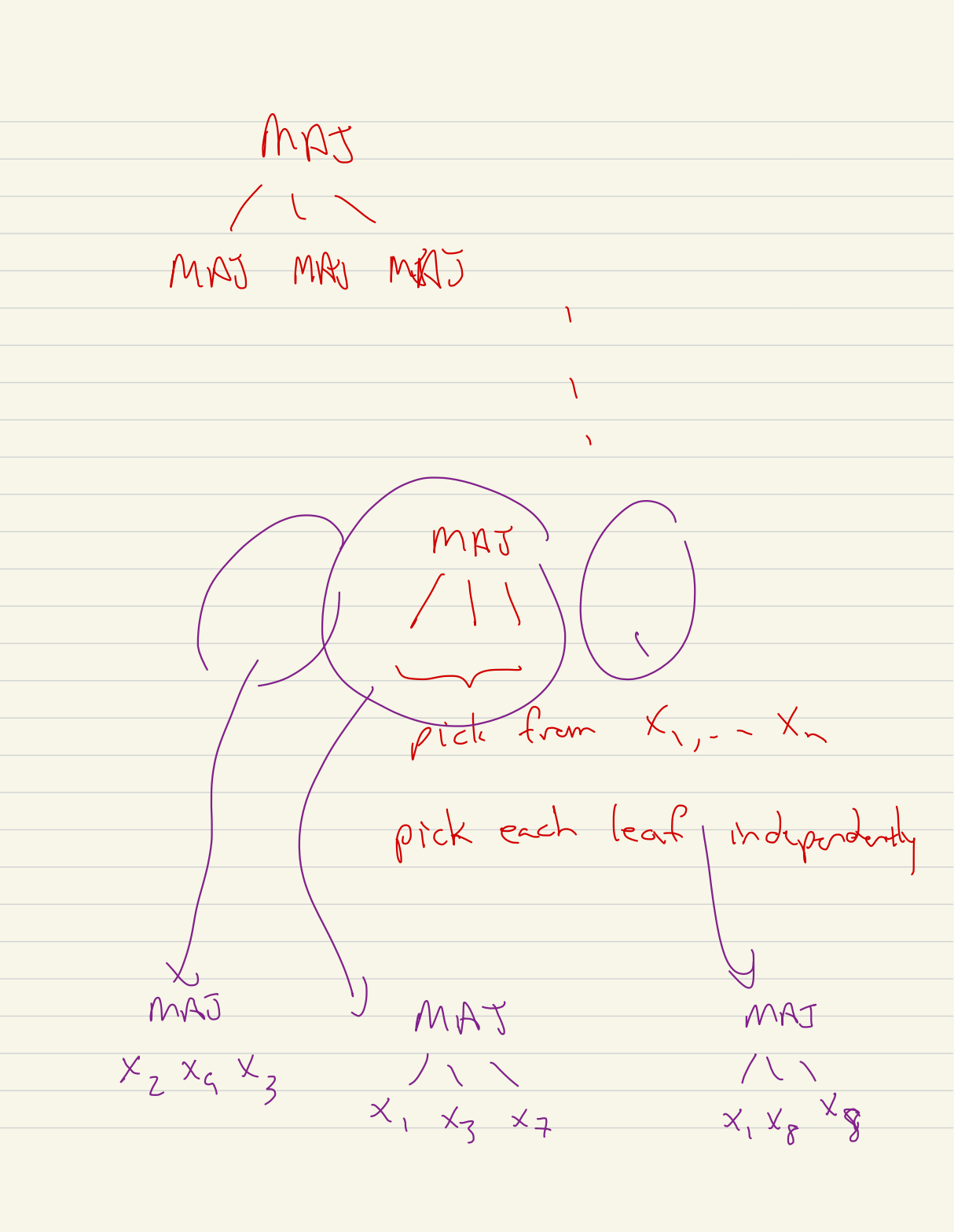
MAJ
/ \
. .

on each fails to compute true MAJ $< \frac{1}{2^n}$

10:24 → 10:29

Break

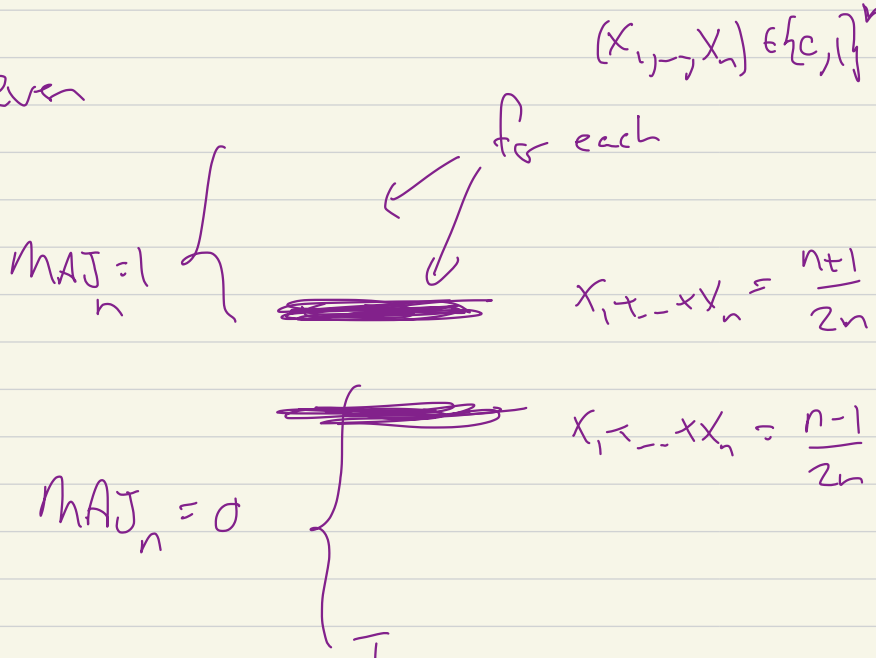
Next week! ICCS, Room 246


MAJ
MAJ MAJ MAJ
MAJ
pick from $x_1, \ldots x_n$
pick each leaf independently
MAJ
x_2 x_4 x_3
MAJ
x_1 x_3 x_7
MAJ
x_1 x_8 x_8

Need:

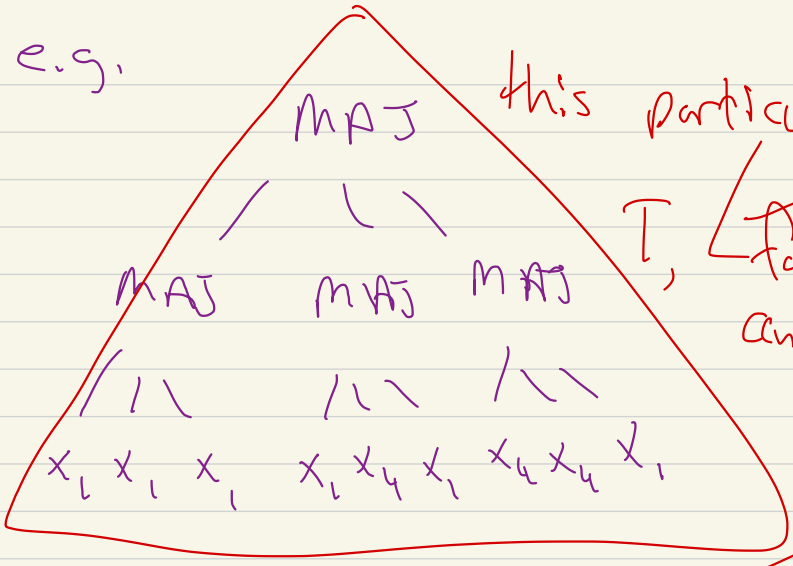
if $x_1, \ldots, x_n$ are set to 0/1 in any of 2^n possible ways

Or even $(x_1, \ldots, x_n) \in \{0,1\}^n$



Say that a MAJ tree, T, is bad on $x_1, \ldots, x_n$ if T does not compute MAJ_n

e.g.



this particular tree, T, fails to compute MAJ_5

$n = 5$

$x_1 = 1,\ x_4 = 1$

$x_2 = x_3 = x_5 = 0$

$MAJ_5(x_1, \ldots, x_5) = 0$

but -

As long as prob for any fixed $x_1, \ldots, x_5$

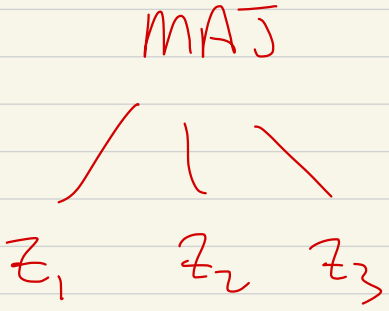
prob of the random tree computing the wrong f_n value $< \frac{1}{2^5}$

$$\text{Prob}\left[\begin{array}{l}\text{random tree computes} \\ \text{the wrong value of} \\ \text{any } X_1, \ldots X_5\end{array}\right] < 1.$$

Size today tree $\simeq n^{6.19\ldots}$

Each X_i appears $\simeq n^{5.19}$ times

Calculation: Say that



Z_1, Z_2, Z_3 independent, and

$$Z_1, Z_2, Z_3 \in \{UBC, SFU\}$$

$$\text{Prob } Z_i = UBC = p$$

$$\text{... } SFU = 1-p$$

What is the prob of

$$MAJ(Z_1, Z_2, Z_3) = UBC$$

p UBC	p UBC	p UBC
(1-p) SFU	(1-p) SFU	(1-p) SFU
Z_1	Z_1	Z_3

Prob that 2 or 3 of

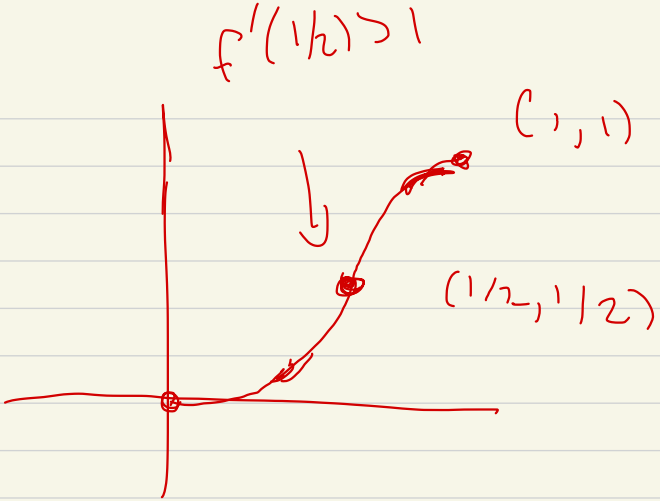
Z_1, Z_2, Z_3 = UBC

3 UBC : p^3

2 UBC, 1 SFU : $3p^2(1-p)$

Total

$$f(p) = p^3 + 3p^2(1-p)$$



$p = 0 \quad f(p) = 0$

$p = 1 \quad f(p) = 1$

$p = 1/2 \quad f(p) = 1/2$

$f(p)$, p small

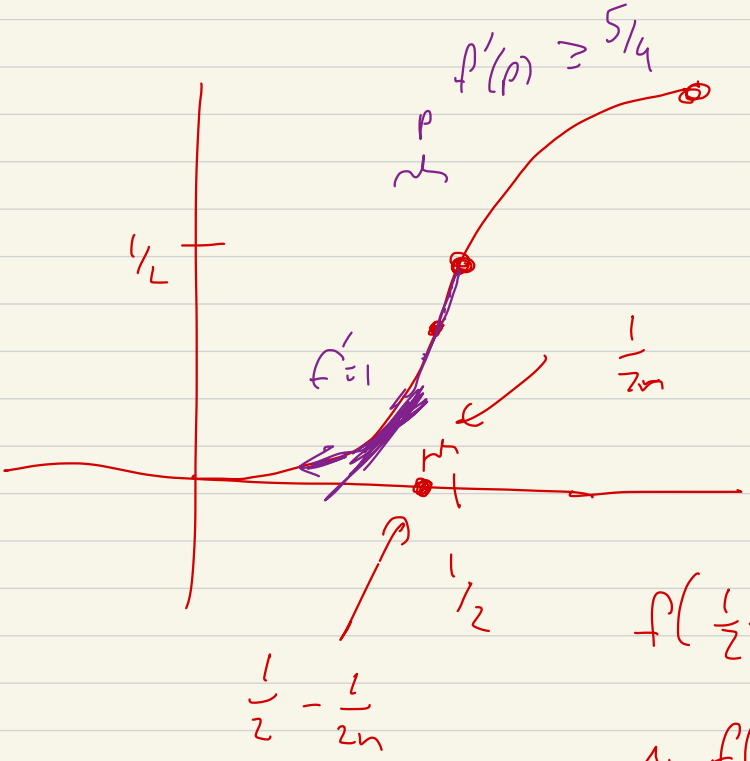
$$p^3 + 3p^2 - 3p^3$$

$$= 3p^2 - 2p^3$$

$$f'(p) = 6p - 6p^2$$

$$f'(1/2) = 6p(1-p) \qquad p = 1/2$$

$$= \frac{6}{4} = \frac{3}{2} > 1$$



$$f\left(\frac{1}{2} - \frac{1}{2n}\right)$$

$$\approx f\left(\frac{1}{2}\right) - \frac{1}{2n}\left(\frac{3}{2}\right)$$

Claim:

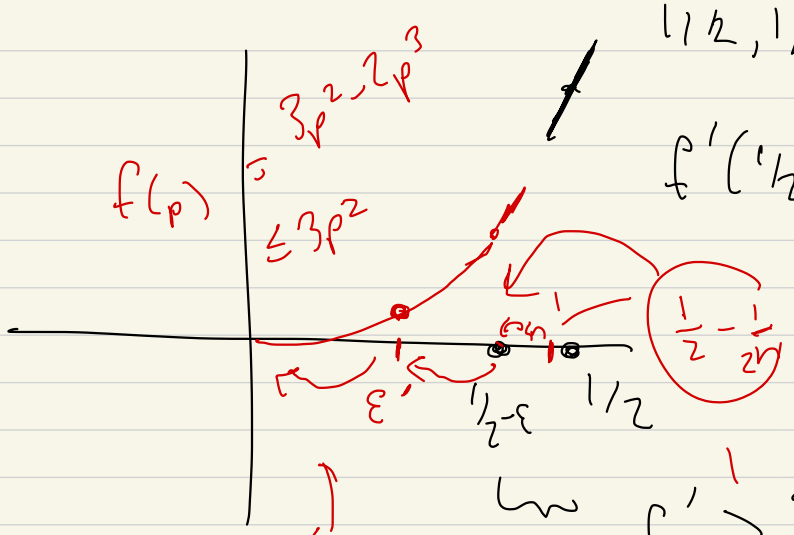
$f^{\text{iterated } r \text{ times}}\left(\frac{1}{2} - \frac{1}{2n}\right)$

$\underbrace{f \circ \cdots \circ f}_{r \text{ times}}\, f(\quad)$

will be $< \frac{1}{2^r}$

after $r \simeq C \cdot \log n + C'$

↑ we'll want to know



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Class Ends