CPSC 536F Jan 27, 2022

- Today: Andre'ev's function giving $\geq c\, n^{1+\gamma}$ lower bound for γ = shrinkage

Spira's Lemma $\left\{ \begin{array}{l} \text{depth vs. size} \\ \text{formulas, circuits} \end{array} \right.$

Start monotone formulas and algebraic formulas.

More "probabilistic method"

Homework:

(Typical in counting Boolean functions, probabilistic methods, etc.)

First Homework theme:

show $$\lim_{n\to\infty} \frac{n!}{\sqrt{n}\,(n/e)^n} = C$$

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(1) Use $f(x) = \log(x)$ monotone increasing:

$$\log(1)+\dots+\log(n-1) \leq \int_1^n \log(t)\,dt \leq \log(2)+\dots+\log(n)$$

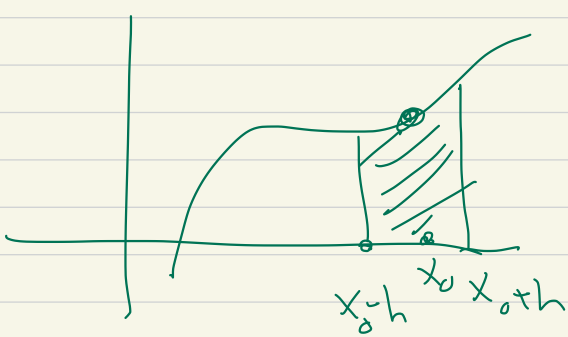
get $\left(\frac{n}{e}\right)^n n^{c_1} \leq n! \leq \left(\frac{n}{e}\right)^n n^{c_2}$

large n.

(2) Show midpoint rule

$$\int_{x_0-h}^{x_0+h} f(t)\,dt$$

$$= \underline{f(x_0)}\, 2h + \text{small}$$



Set

$$F(h) = \left(\int_{x_0-h}^{x_0+h} f(t)\,dt \right) - 2h\, f(x_0)$$

Assume f is twice cont differentiable

so that

$F(0)$, $F'(0)$, $F''(0)$, and

$$F'''(h) = f''(x_0-h) + f''(x_0+h)$$

Deduce Taylor's thm

$$F(h) = \frac{1}{6} h^3 \cdot F'''(\xi)$$

for some $\xi \in [0, h]$ $(h > 0)$

Use

$$\int_{1/2}^{n+\frac{1}{2}} \log(t)\, dt$$

$$= \int_{1/2}^{3/2} + \int_{3/2}^{5/2} + \cdots + \int_{n-1/2}^{n+1/2}$$

Use midpoint rule

$$\underline{\sqrt{n}\left(\frac{n}{e}\right)^n} c_1' \leq \underline{n!} \leq \underline{\sqrt{n}\left(\frac{n}{e}\right)^n} c_1$$

$c_1', c_1 > 0$ constants

- Show that

$$g(n) = \frac{n!}{\sqrt{n}\,(n/e)^n}$$

that

$$\lim_{n\to\infty} g(n) \text{ exists !}$$

(1) Show $\dfrac{g(n+1)}{g(n)} = \boxed{1 + O\left(\frac{1}{n^2}\right)}$

(2) Show that for any C constant

$$(1+C)\left(1+\frac{C}{2^2}\right)\left(1+\frac{C}{3^2}\right)\left(1+\frac{C}{4^2}\right)\cdots$$

this has a finite limit.

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Rough idea:

$(1+\varepsilon_1)(1+\varepsilon_2)\cdots$

$\varepsilon_i \to 0$ as $i \to \infty$

then can use

$$\log_e(1+x) = x + O(x^2)$$

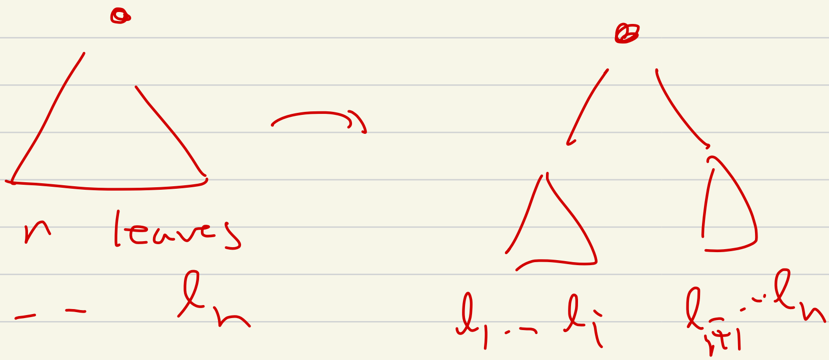
if say $-\frac{1}{2} \leq x \leq \frac{1}{2}$.

$$\log(\quad) = \varepsilon_1 + \varepsilon_2 + \cdots + O(\varepsilon_1^2 + \varepsilon_2^2 + \cdots)$$

Homework:

If $T(n) = \#$ binary trees on n leaves

count



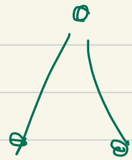
$$T(n) = \sum_{i=1}^{n-1} T(i)\, T(n-i)$$

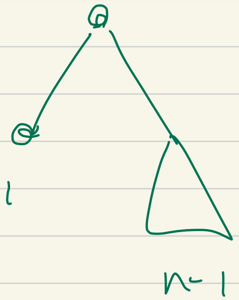
here: $T(1) = 1$

binary tree?

one vertex tree ℓ_1

$T(2) = T(1)\,T(1) = 1$





$\rightarrow \quad 1 \cdot T(n-1)$

$=$

Involves generating functions!

Set $\displaystyle\sum_{n=1}^{\infty} z^n\, T(n)$

$= z \cdot 1 + z^2 \cdot 1 + z^3\, 2 + \dots$

$= G(z)$

Note!

$$\left(\sum z^n T(n)\right)\left(\sum z^n T(n)\right)$$

$$\left(zT(1)+z^2T(2)+\dots\right)\left(zT(1)+z^2T(2)+\dots\right)$$

$$= z^2\, T(1)T(1)$$

$$+\; z^3\left(T(1)T(2)+T(2)T(1)\right)$$

$$+\; \vdots$$

$$+\; z^n\left(\qquad\qquad\right)$$

$$= z^2T(2)+z^3T(3)+\dots = G(z)-z$$

Formally

$$G^2 = G - z$$

$$G^2 - G + z = 0$$

$$G = \frac{1 \pm \sqrt{1-4z}}{2}$$

gives formula for $T(n)$.

(Use fact!

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n$$

then if $|a_n|^{1/n} = O(1)$

$|b_n|^{1/n} = O(1)$

then

$f(z), g(z)$ are defined for

$$|z| < \frac{1}{C}$$

$|a_n|^{1/n}, |b_n|^{1/n} \leq C$ n large

and

$$f(z) = g(z)$$

iff $a_i = b_i$ for all i.

Tools using $\log(1+x) = x + O(x^2)$

for $|x| \le 1/2$, etc.

used in probabilistic methods.

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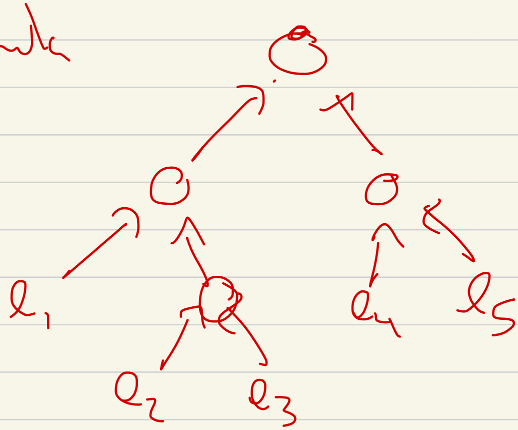
Last time:

Subbotovskaya (1961)

De Morgan formula



Gates $\circ$

are $\wedge$, $\vee$

$\ell_i = x_1, \dots, x_n, \bar{x}_1, \dots, \bar{x}_n$

Randomly pick one $x_1, \ldots, x_n$

set it to $0, 1$ prob $1/2, 1/2$

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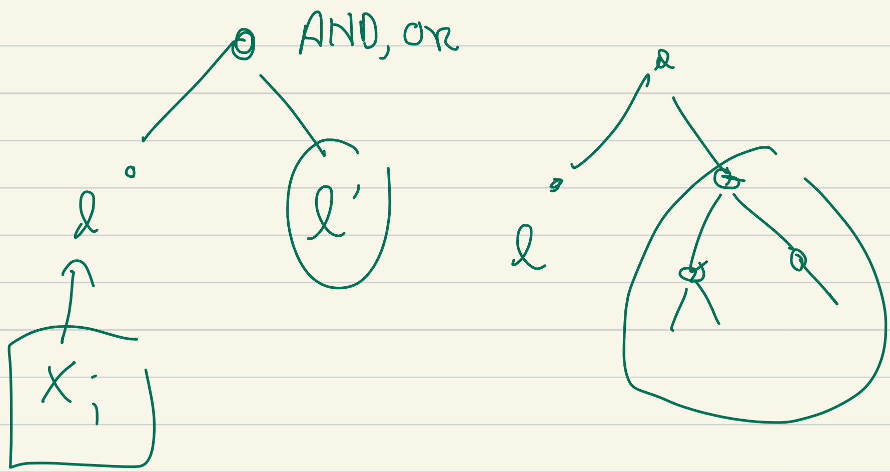
Then you repeat this again,

repeat until m variables left



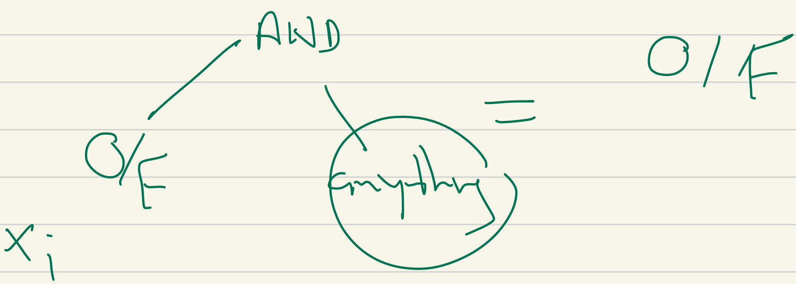
If you pick x_i to set to $0, 1$

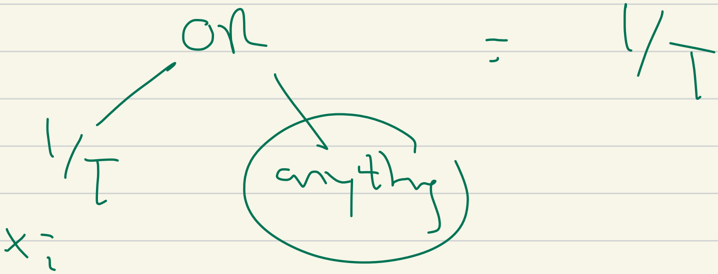
Start with L leaves,

Remaining # leaves

or average after first step

$$\text{Avg leaves} \le L\left(1 - \frac{3/2}{n}\right)$$





If done until m leaves left

Avg leaves left $\leq L\left(1-\frac{3/2}{n}\right)\left(1-\frac{3/2}{n-1}\right)$

$\prod \left(1-\frac{3/2}{m}\right)$

take log, sum, estimate

$\leq L \frac{1}{n^{3/2}} m^{3/2}$

Last time: Parity!

$f(x_1, \ldots, x_n) = x_1 \oplus \cdots \oplus x_n$

then after setting

$n - m$ variables to $0/1$

still got $\tilde{f}(\quad) = \text{parity}(\quad)$

m variables

$\Rightarrow$ $L(n)$ = size min formula

to compute

$\text{parity}(x_1, \ldots, x_n)$

$Cn^2 \geq L(n) \geq n^{3/2} \cdot C$

Question:

Goal is to find

$f(x_1, \ldots, x_n)$ for sequence n's tending to ∞

s.t.

$$L = L\left(f(x_1, \ldots, x_n)\right)$$

(1) f is in NP

even f has any poly time algorithm to compute it

(2) We can prove $L(f(x_1, \ldots, x_n))$

is $\geq$ something large = larger than poly(n) ?

Best such bound:

$$L(\text{ Andre'ev's function }) \geq n^{1+\gamma}$$

If under 0/1 restrictions of all but m of f's variables

(expected avg) (size formula that remains)

$$\leq L\left(\frac{m}{n}\right)^{\gamma}$$

Subbotovskaya shows $\gamma \geq 3/2$

Best possible $\gamma \leq 2$

since

Parity n vars,
n is a power of 2 $\lesssim n^2$

$\Rightarrow$

Parity $\geq c\, n^{\gamma}$

Andreev $\geq c\, n^{1+\gamma}$

Andre'ev: N, ℓ parameters

$x_1, \ldots, x_{2^N}$ describe all functions $\{0,1\}^N \to \{0,1\}$

$$z_1^1 \ldots z_N^1 = \vec{z}^1$$

$$z_1^2 \ldots z_N^2 = \vec{z}^2$$

$$\vdots \qquad \vdots$$

$$z_1^\ell \ldots z_N^\ell = \vec{z}^\ell$$

View

$$x_1, \dots, x_{2^N}$$

as describing $f : \{0,1\}^N \to \{0,1\}$

Andreev function

$$(x_1, \dots, x_{2^N}, \vec{z}^{\,1}, \dots, \vec{z}^{\,\ell})$$

$$= \quad f(\vec{z}^{\,1} \oplus \dots \oplus \vec{z}^{\,\ell})$$

n variables! $\quad 2^N = n/2$

$$\ell N = n/2$$

if $\ell = \dfrac{n}{2N}$ is an integer

N = power of 2

Assume N = power of 2

$n = 2^N$, $\ell = \dfrac{n}{2N} = \dfrac{n}{2\log_2 n}$

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Claim: If you randomly restrict

f by eliminating all but

$L(\ )$ $\underset{\text{restrict } n-m}{\longrightarrow}$ $\underset{\text{size}}{\text{leaves}}$ $L \leq \underset{L}{\text{original}} \left(\dfrac{m}{n}\right)^{\gamma}$

Break 10:25 – 10:30

Spira's Lemma!

Formula: size, $\boxed{\text{depth}}$

Circuits: size, $\boxed{\text{depth}}$

very difficult

good estimates for monotone functions

Spira's Lemma:

If f has a DeMorgan formula of size L, then it has a formula of depth $\leq C \underline{\log_2 L}$, C = universal.

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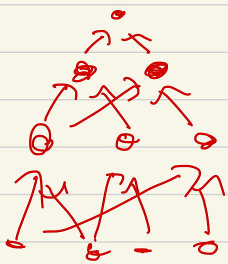
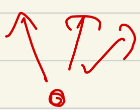
Circuit!

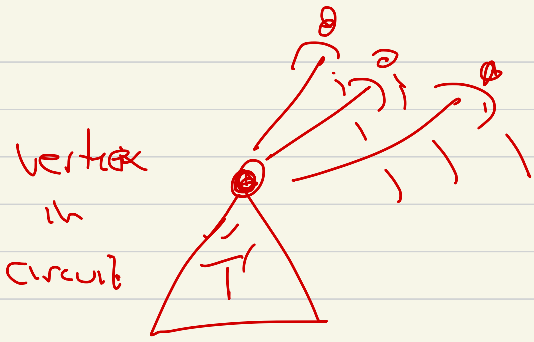


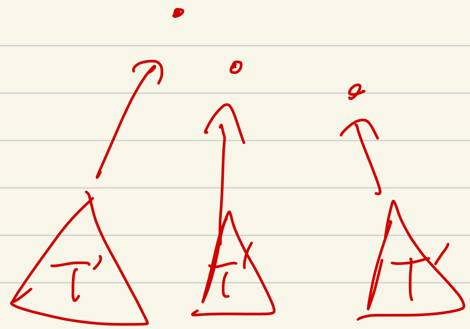
Equivalent Depth formula!



outdegree ≥ 1

Depth := longest path from leaf to root





"unfold circuit"

$\Rightarrow$

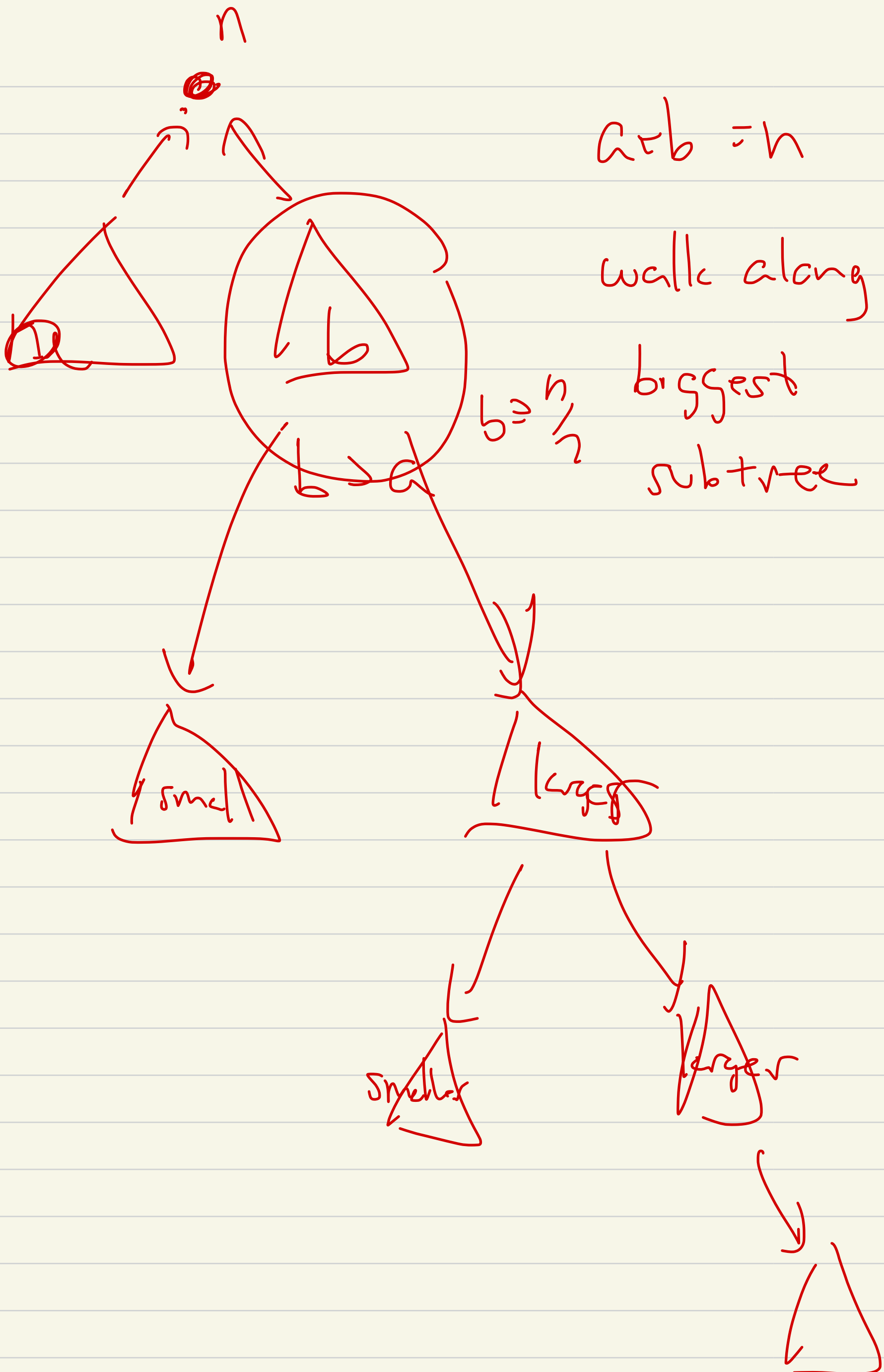
circuit depth = formula depth

given same gates

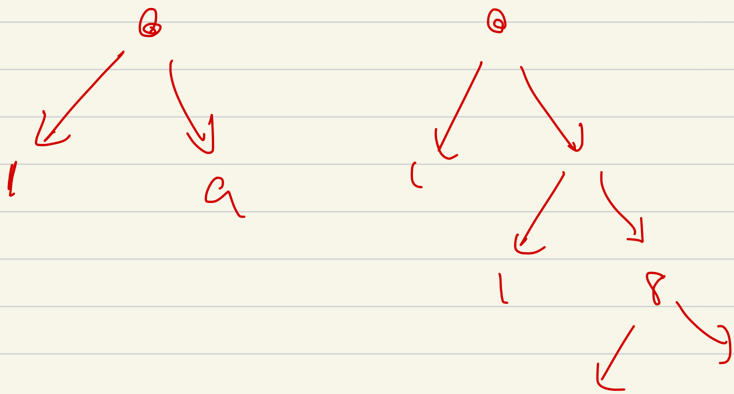
$\log_2 L \le$ formula depth $\le C \cdot \log_2 L$

$a + b = n$

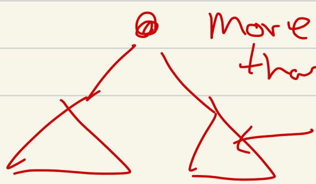
walk along biggest subtree



Observation! In any n tree on n leaves, there is always a subtree on $\geq \frac{n}{3}$, $\leq \frac{2n}{3}$ leaves



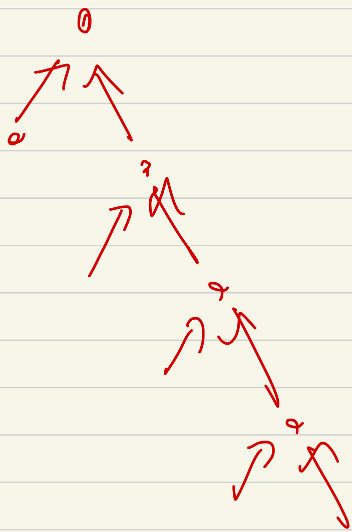
first vertex in following largest subtree



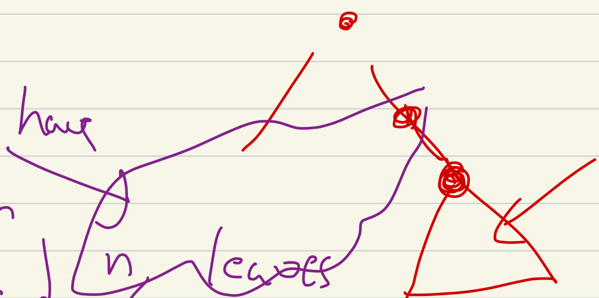
more than $\frac{2n}{3}$ leaves

larger has $\geq \frac{n}{3}$ leaves

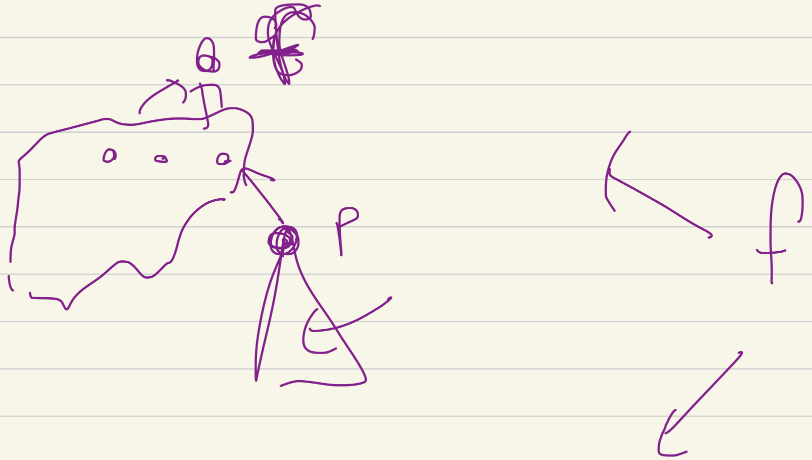
Idea: Could have formula size ~~L~~ and depth $L-1$



at some point



Then you rebalance

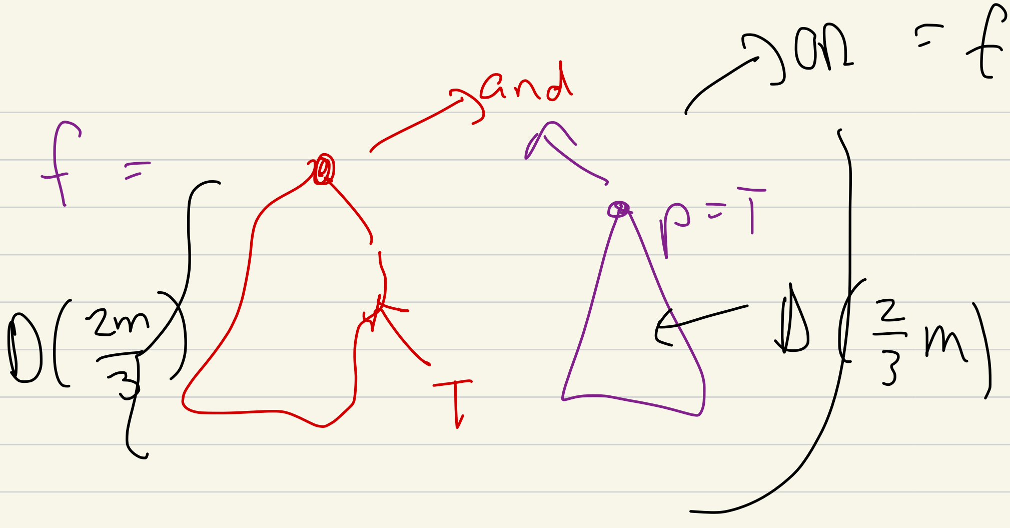


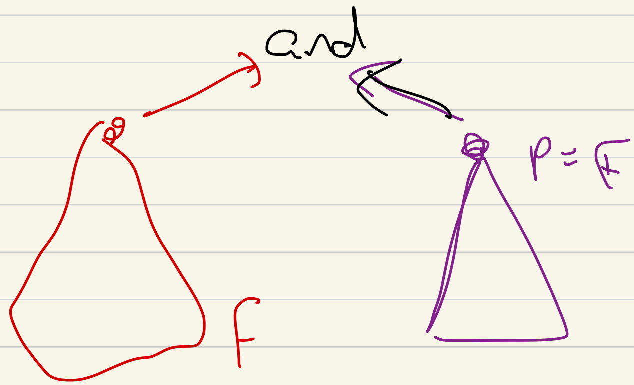
= either

$p = T$ and

or

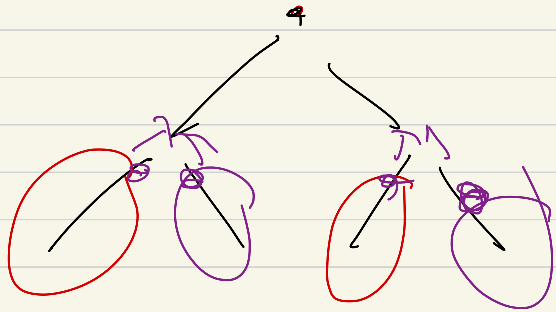
$p = F$ and





continue!

$D(m)$ = max depth needed to compute a formula size m



depth 2

$D\left(\frac{2m}{3}\right)$

$$D(m) \leq 2 + D\left(\frac{2m}{3}\right)$$

$$\leq 2+2+D\left(\left(\frac{2}{3}\right)^2 m\right)$$

$$\leq 2+2+2+D\left(\left(\frac{2}{3}\right)^3 m\right)$$

$$\leq \dots$$

$$2r + D\left(\left(\frac{2}{3}\right)^r m\right)$$

$$\left(\frac{2}{3}\right)^r m \le 1$$

$$\log(\qquad) \le 0$$

$$r \log \frac{2}{3} + \log m \le 0$$

r at least $\dfrac{\log m}{\log 3/2}$

$$D(m) \le 2 \frac{\log m}{\log 3/2} \le c(\log m)$$

Class Ends