CPSC 536F, 2022: HOMEWORK SET 2

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You may work on homework in groups, **but you must write up your own solutions individually and must acknowledge with whom you worked.** You must also acknowledge any sources you have used beyond the textbooks and other course references. You must use the notation we use in class and/or the course references.

(1) Recall the cartesian product $G \times H$ of graphs G and H (see class notes of March 1 and 6). Let (1) G be a graph on n vertices, (2) $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A_G (the adjacency matrix of G), and (3) $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be a corresponding (real) orthonormal eigenbasis for A_G (i.e., $A\mathbf{x}_i = \lambda_i \mathbf{x}_i$, and $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are orthonormal; hence each $\mathbf{x}_i$ is a function $V_G \to \mathbb{R}$). Similarly let (1) H be a graph on m vertices, (2) $\nu_1, \ldots, \nu_m$ be the eigenvalues of A_H, and (3) $\mathbf{y}_1, \ldots, \mathbf{y}_m$ a corresponding eigenbasis.

(a) Show that the vectors $\mathbf{x}_i \otimes \mathbf{y}_j$ with $1 \le i \le n$ and $1 \le j \le m$ form an orthonormal set of vectors $V_G \times V_H \to \mathbb{R}$.

(b) Show that

$$A_{G \times H}(\mathbf{x}_i \otimes \mathbf{y}_j) = (\lambda_i + \nu_j)(\mathbf{x}_i \otimes \mathbf{y}_j).$$

(c) Conclude that $\lambda_i + \nu_j$ with $1 \le i \le n$ and $1 \le j \le m$ are the eigenvalues of $A_{G \times H}$.

(2) Show that if A is a subset of $\{1, \ldots, n\}$ and $\chi_A \in \mathbb{R}^n$ is the vector whose components are 1 on the elements of A, and 0 elsewhere (i.e., χ_A is the characteristic vector o A), then the projection of χ_A onto the orthogonal complement of the constant vector $(1, \ldots, 1)$ has (L^2-) norm

$$\sqrt{\frac{|A|\,(n - |A|)}{n}}.$$

(3) Let (1) G be a d-regular graph, (2) A_G be its adjacency matrix, (3) $\lambda_1 \ge \ldots \ge \lambda_n$ the eigenvalues of A_G, and (4) $\rho = \max_{i \ge 2} |\lambda_i|$. This exercise gives a quick lower bound on ρ.

(a) Show that $\mathrm{Trace}(A_G^2) \ge nd$.

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(b) Use the fact that $\mathrm{Trace}(A_G^2) = \lambda_1^2 + \cdots + \lambda_n^2$ to show that $\rho^2 \geq (dn - d^2)/(n-1)$.

(4) Recall the diameter bound (class, March 10) of $\log(n)/\log(d/\rho)$ for a d-regular graph where ρ is the bound on the absolute values of all but the first eigenvalue.
  (a) Let G be the N-dimensional Boolean hypercube, with $n = 2^N$ vertices. What is ρ? Explain why the above diameter bound is useless here.
  (b) Let $A = A_G + I$ where I is the identity matrix. Hence the first eigenvalue of A is $N+1$. Explain why if ρ' is the absolute value of all but the eigenvalue $N+1$, then $\log(n)/\log((N+1)/\rho')$ gives a bound on the diameter.
  (c) How does the previous bound compare with the true diameter of the hypercube?

(5) Let H be the graph with 3 vertices, $\{v_1, v_2, v_3\}$, with no self-loops and (1) one edge between v_1 and v_2, (2) one edge between v_1 and v_3, and (3) two edges between v_2 and v_3. Let $\mu\colon H \to H$ an automorphism of H that takes v_1 to itself and exchanges v_2 and v_3. (See class, March 22.) Compute the eigenpairs (i.e., eigenvalue/eigenvector pair) of the even functions, i.e., those functions $f\colon V_H \to \mathbb{R}$ such that $f\mu = f$. Show that along with the eigenpairs of the odd functions give all the eigenpairs of A_H.

(6) Show that if $\pi\colon H \to G$ is an étale map of finite graphs, then there is a graph H' that contains H as a subgraph, and a covering map $\pi'\colon H' \to G$ such that π equals the inclusion map $H \to H'$ following by the covering map $H' \to G$. [Hint: If the sets $\pi_V^{-1} v$ as v varies over V_G are of different sizes, add isolated vertices to H to obtain a graph $\tilde{H}$ and an extension of π, $\tilde{\pi}\colon \tilde{H} \to G$ such that $\tilde{\pi}^{-1}(v)$ are all of the same size. Then explain how to add edges to $\tilde{H}$ to obtain the desired graph H' and covering map π'.]

(7) Use the 5-to-1 covering Galois map from the Petersen graph to the graph on two vertices given in class (March 24) to compute $\rho = \max_{2 \leq i \leq 10} |\lambda_i|$, either exactly or approximately.

(8) Show that if π is the 10-to-1 covering map from the Petersen graph, G, to, B, graph of one vertex (given on March 24, where two directed edges are inverses of each other, and one edge is its own inverse), then $\mathrm{Aut}_B(G)$ is only of order 5, and hence π is not Galois. In particular, if A, B, C, D, E are the outer vertices of the Petersen graph, and a, b, c, d, e are the inner vertices (see class, March 24), then there is no element of $\mathrm{Aut}_B(G)$ that takes A to any of the inner vertices, but these is an element taking A to any of the outer vertices.

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