CPSC 536F Nov 17, 2025

- Last time: if $\vec{v} \perp \vec{u}$, then

$$\mathcal{R}_A(\vec{u}+\varepsilon\vec{v}) = \mathcal{R}_A(\vec{u}) + O(\varepsilon^2)$$
$$+ \varepsilon\, 2\,(A\vec{u})\cdot\vec{v}$$

Hence $\vec{u}$ with $\|\vec{u}\|_{L^2}=1$ maximizes

$\mathcal{R}_A(\vec{u}) \Rightarrow \forall \vec{v} \perp \vec{u}$, $\vec{v} \perp A\vec{u}$

Now: min-max, max-min, etc.

Slick proof:

Lemma: $\forall A, B$, $\{ x \in \mathbb{R} \mid A + Bx$ has multiple (algebraic) eigenvalues $\}$

is either:

(1) finitely many $x \in \mathbb{R}$, OR

(2) all $x \in \mathbb{R}$

=

SVD:

=

Reversible Markov chains

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Other apps ...

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Kernels, Heat Equation

Last time: if $\vec{v} \perp \vec{u}$, then

$$R_A(\vec{u}) = \frac{(A\vec{u})\cdot\vec{u}}{\vec{u}\cdot\vec{u}}$$

$$\max_{\vec{u}\neq\vec{0}} R_A(\vec{u}) = \max\Big\{(A\vec{u})\cdot\vec{u} \;\Big|\; \underbrace{\vec{u}\cdot\vec{u}=1}_{S^{n-1}}\Big\}$$

$$= \max\Big\{(A\vec{u})\cdot\vec{u} \;\Big|\; \vec{u}\in S^{n-1}\Big\}$$

$$(\vec{u}+\varepsilon\vec{v})\cdot(\vec{u}+\varepsilon\vec{v}) = \vec{u}\cdot\vec{u} + O(\varepsilon^2)$$

$$R_A(\vec{u}+\varepsilon\vec{v}) = R_A(\vec{u}) + O(\varepsilon^2)$$

$$+\ \varepsilon\ 2\,(A\vec{u})\cdot\vec{v}$$

$1^{\underline{st}}$ Observation:

Let $\vec{u}_1 \in \mathbb{S}^{n-1}$ at which

$R_A(\vec{u})$ is maximized:

Then $\forall \vec{v} \perp \vec{u}_1$,

$R_A(\vec{u}_1 + \varepsilon \vec{v})$ has no order ε term

$$= \varepsilon^0 \_\_\_ + \underbrace{\varepsilon^1 \_\_\_}_{\text{has to be } 0} + O(\varepsilon^2)$$

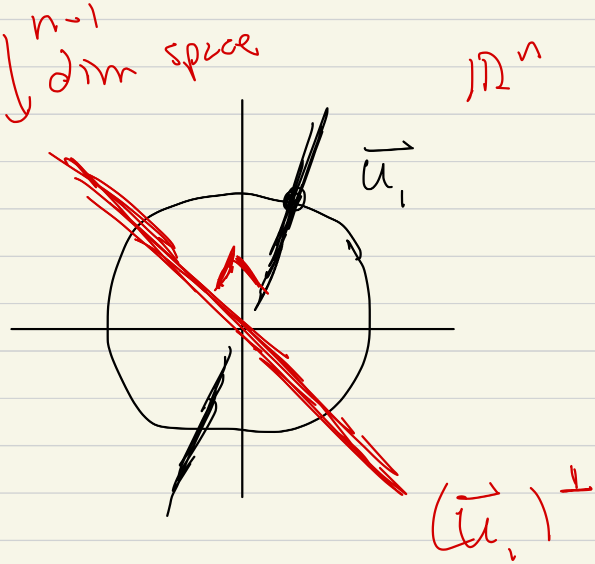
$\Rightarrow 2(A\vec{u}_1) \cdot \vec{v} = 0 \quad \forall \vec{v} \perp \vec{u}_1$

$\Rightarrow A\vec{u}_1 = \lambda_1 \vec{u}_1$ for some $\lambda_1 \in \mathbb{R}$

Note

$$R_A(\vec{u}_1) = \frac{(A\vec{u}_1)\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1} = \frac{\lambda_1\vec{u}_1\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}$$

$$= \lambda_1$$



$$A\vec{u}_1 = \lambda_1\vec{u}_1$$

<u>2nd Step</u>: Next, let $\vec{u}_2$ be

where $R_A(\vec{u})$ is maximized

subject to those $\vec{u} \perp \vec{u}_1$

Now let $\vec{v} \perp \vec{u}_2, \vec{u}_1$

$$R_A(\vec{u}_2 + \varepsilon\vec{v}) = R_A(\vec{u}_2) + \varepsilon\,(A\vec{u}_2)\cdot\vec{v} + O(\varepsilon^2)$$

so

① If $\vec{v} \perp \vec{u}_1, \vec{u}_2$, then

$$(A\vec{u}_2)\cdot\vec{v} = 0$$

2-dim
plane in $\mathbb{R}^n$

So $A\vec{u}_2 = \lambda_2 \vec{u}_2 + \alpha \vec{u}_1$

But: $\vec{u}_2 \perp \vec{u}_1$, hence

$$\underbrace{(A\vec{u}_2)\cdot\vec{u}_1}_{} = (\lambda_2\vec{u}_2 + \alpha\vec{u}_1)\cdot\vec{u}_1$$

$$= (\lambda_2\vec{u}_2\cdot\vec{u}_1) + \alpha\,\underbrace{\vec{u}_1\cdot\vec{u}_1}$$

$$= \underbrace{0 + \alpha\cdot 1}_{\alpha}$$

$$(\vec{u}_2\cdot(A^T\vec{u}_1))$$

$$\downarrow A = A^T$$

So $\alpha = 0$

$$\vec{u}_2\cdot(A\vec{u}_1) = \vec{u}_2\cdot(\lambda_1\vec{u}_1) = \underbrace{0}$$

So $\quad A\vec{u}_2 = \lambda_2 \vec{u}_2$ (eigenvector)

But now

$$R_A(\vec{u}_2) = \frac{A\vec{u}_2 \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} = \lambda_2$$

but

R_A maximised everywhere at $\vec{u}_1$,

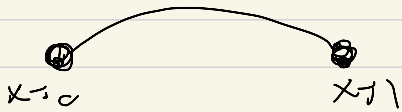
max value is λ_1

R_A s.t. some condition (being $\perp \vec{u}_1$) max at $\vec{u}_2$

$$R_A(\vec{u}_2) \leq R_A(\vec{u}_1)$$

$$\lambda_2 \leq \lambda_1$$

Rem: Solving Poisson eq.,
heat equation on $(0,1) \subset \mathbb{R}$



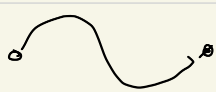
$$-\Delta f_1 = \lambda_1 f_1 , \qquad f_1(0) = f_1(1) = 0$$

$$\rightsquigarrow \quad f_1(x) = \frac{1}{\sqrt{2}} \sin(\pi x)$$

$$\lambda_1 = \pi^2$$

Then max $R_{(-\Delta)}(f)$

s.t. $f(0) = f(1) = 0$ and $f \perp f_1$

get f_2 

So even for ∞-dimensional

$$-\Delta = -\left(\frac{d}{dx}\right)^2 \quad \text{on} \quad (0,1) \subset \mathbb{R}$$

same method works...

Consequence: If $A \in \mathbb{R}^{n \times n}$,

$A^T = A$, then

$$A\vec{u}_1 = \lambda_1 \vec{u}_1, \quad A\vec{u}_2 = \lambda_2 \vec{u}_2, \ldots$$

with
$$\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$$A\left[\vec{u}_1 \middle| \vec{u}_2 \middle| \cdots \vec{u}_n\right]$$

$$= \left[A\vec{u}_1 \middle| A\vec{u}_2 \middle| \cdots A\vec{u}_n\right]$$

$$= \left[\lambda_1\vec{u}_1 \middle| \lambda_2\vec{u}_2 \middle| \cdots \middle| \lambda_n\vec{u}_n\right]$$

$$= \left[\vec{u}_1 \middle| \vec{u}_2 \middle| \cdots \middle| \vec{u}_n\right] \begin{bmatrix} \lambda_1 & & 0\text{'s} \\ & \lambda_2 & \\ 0\text{'s} & & \ddots & \lambda_n \end{bmatrix}$$

$$AU = U \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

and

$$U^T U = \begin{bmatrix} - & \vec{u}_1^T & - \\ - & \vec{u}_2^T & - \\ & \vdots & \\ - & \vec{u}_n^T & - \end{bmatrix} \begin{bmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} \vec{u}_1 \cdot \vec{u}_1 & \vec{u}_1 \cdot \vec{u}_2 & \cdots \\ & & \\ & & \end{bmatrix} = I$$

So $U \in \mathbb{R}^{n \times n}$ s.t. $U^T U = I$ is called an orthogonal matrix

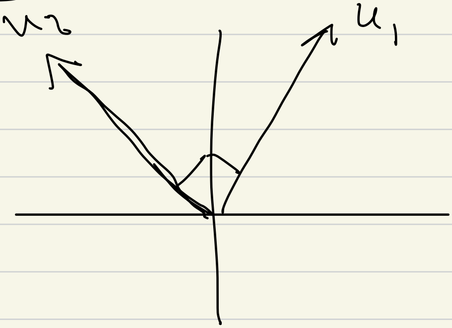
and

$$AU = U\Lambda, \quad \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$A = U \Lambda U^{-1} \qquad \left(\begin{array}{c} U^T U = I \\ U^{-1} = U^T \end{array} \right)$$

$$= U \Lambda U^T$$

$\vec{u}_2$ $\vec{u}_1$ $\mathbb{R}^2$



$\vec{x} \longmapsto U\vec{x}$ this map preserves dot products

Claim: Let

$$A = \Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

be (symmetric) and diagonal matrix...

$$\lambda_1 = \max_{\vec{u} \neq 0} R_A(\vec{u})$$

$$= \max_{\substack{\vec{V} \subset \mathbb{R}^n \\ \dim(\vec{V}) = 1}} \left(\min_{\substack{\vec{v} \in \vec{V} \\ \vec{v} \neq 0}} R_A(\vec{v}) \right)$$

$$\lambda_2 = \max_{\substack{\mathcal{V} \subset \mathbb{R}^n \\ \dim(\mathcal{V}) = 2}} \left(\min_{\substack{\vec{v} \in \mathcal{V} \\ \vec{v} \neq 0}} R_A(\vec{v}) \right)$$

this maximum occurs at

$$\mathcal{V} = \operatorname{span}(\vec{u}_1, \vec{u}_2)$$

Class Ends

$-\Delta$: $\quad \Omega \subset \mathbb{R}^n$, Ω domain (bounded, open)

$\overline{\Omega}$ compact

$$\boxed{\int (-\Delta u)\cdot u \, dV}$$

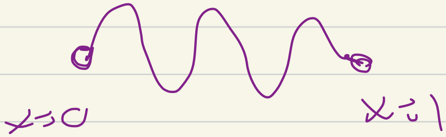
$$= \int (\nabla u)\cdot(\nabla u)\, dV \qquad u \in C_c^\infty(\Omega)$$

$$\geq 0$$

Ω

Weyl's Law

$$\lambda_n \sim n^{2/d} \cdot (\mathrm{Vol}(\Omega))^{2} \text{ const}$$



$$f_n = \sin(\pi n x)$$

λ_n of $-\Delta$ is $\pi^2 n^2$

$$\int_0^1 (f'(x))^2 \, dx = \text{Dirichlet}(u)$$

$f \in C_c^\infty(\Omega)$:

$R_{-\Delta}(f)$:

$f_1, f_2, \dots \in C_c^\infty(\Omega)$

s.t.

$$\lim_{n\to\infty} R_{-\Delta}(f_n) = \inf_{\substack{f\neq 0 \\ f\in C_c^\infty(\Omega) \\ \int f^2 = 1}} \left(R_\Delta(f)\right)$$

$\int f^2 = 1$

$f_n \longrightarrow$ limit ??

look for f s.t.

$\forall g \in C_c^\infty(\Omega)$

$$\int g f_n \longrightarrow \int g f$$

if so

$$f_n \xrightarrow{\text{weakly}} f$$

$0 \le \int f^2 \le 1$, $\quad \int (f_n)^2 \to \inf$

Avner Friedman, PDEs