CPSC 536F Nov 14, 2025

$A \in \mathbb{R}^{n\times n}$, $A^T = A$, has eigenvalues

$$\lambda_n \leq \dots \leq \lambda_2 \leq \lambda_1$$

(with ON set of eigenvectors, ...)

[rem: for graph Laplacians, often write $\lambda_1(\Delta_G) \leq \lambda_2(\Delta_G) \leq \dots$]

[idea: "most important eigenvalues" should be λ_1 and/or λ_2]

Proof: Let $R_A(\vec{u})$ be maximized at $\vec{u}_1$; then $\lambda_1 = R_A(\vec{u}_1)$ and

$$A\vec{u}_1 = \lambda_1 \vec{u}_1$$

Min-max and max-min.

=

SVD! $A \in \mathbb{R}^{m \times n}$. Then

$$\lambda_1(A^TA) \geq \lambda_2(A^TA) \geq \cdots$$

are written $\sqrt{\lambda_i(A^TA)} = \underbrace{\sigma_i(A)}_{\text{singular values of } A}$

and

$$A = \underset{m \times n}{U} \ \underset{\substack{m \times n \\ \text{diagonal}}}{\Sigma} \ V^T$$

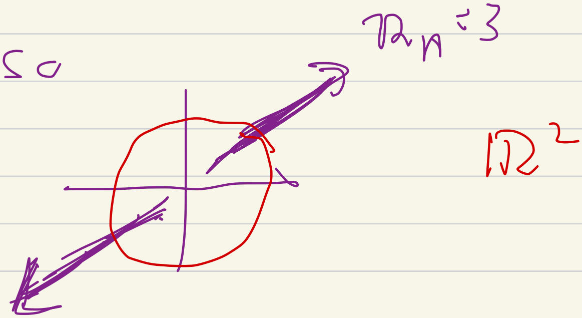
Also: the absolutely slickest proof of the spectral theorem, etc. (for finite matrices)

$$R_A(\vec{u}) = \frac{(A\vec{u})\cdot\vec{u}}{\vec{u}\cdot\vec{u}}$$

$\vec{u} \neq \vec{0}$, and $R_A(\vec{u}) = R_A(\vec{u}\alpha)$

$\alpha \in \mathbb{R}\setminus\{0\}$

So



(1) A maximum of R_A on S^{n-1}

exists . . . .

take any $\vec{v} \perp \vec{u}$ (trick, to simplify calculation)

then

$$R_A(\vec{u} + \varepsilon \vec{v})$$ ← the ε term must vanish

$$f(\varepsilon) = R_A(\vec{u} + \varepsilon \vec{v})$$

then

$f'(0)$ must vanish

$$(\vec{u} + \varepsilon \vec{v}) \cdot (\vec{u} + \varepsilon \vec{v}) = \vec{u} \cdot \vec{u} + 2\varepsilon\, \vec{u} \cdot \vec{v} + \varepsilon^2\, \vec{v} \cdot \vec{v}$$

$$= \vec{u}\cdot\vec{u} + 0\cdot\varepsilon + (\vec{v}\cdot\vec{v})\,\varepsilon^2$$

$$R_A(\vec{u}+\varepsilon\vec{v}) = \frac{\left(A(\vec{u}+\varepsilon\vec{v})\right)\cdot(\vec{u}+\varepsilon\vec{v})}{(\vec{u}+\varepsilon\vec{v})\cdot(\vec{u}+\varepsilon\vec{v})}$$

$$\frac{1}{(\vec{u}+\varepsilon\vec{v})\cdot(\vec{u}+\varepsilon\vec{v})} = \frac{1}{\vec{u}\cdot\vec{u} + \varepsilon^2 \underbrace{O(1)}_{\vec{u},\,\vec{v}\ \text{fixed}}}$$

$$= \frac{1}{\vec{u}\cdot\vec{u}}\ \frac{1}{\left(1+\varepsilon^2 O(1)\right)}$$

$\left[\ O(1) = \text{order}\ \right]$

$$\frac{1}{\vec{u}\cdot\vec{u}}\Big(1+\varepsilon^2\underbrace{O(1)}_{\text{different }O(1)\text{ than before}}\Big)$$

$$\frac{1}{\vec{u}\cdot\vec{v}}\Big(1-\varepsilon^2\,O(1)\Big)$$

$$\frac{1}{1+4\varepsilon^2}=1-(4\varepsilon^2)+(4\varepsilon^2)^2+\_\_\_$$

Recall: If $\vec{u},\vec{v}$ fixed, then

$O(1)$ any function, f, s.t.

for $|\varepsilon|$ suff small, $|f(\varepsilon)|\leq C_{\vec{u},\vec{v}}$

Rem in C.S., $n \to \infty$

(e.g. size of input)

$$O(\underline{n}^3) = \left\{ \text{all functions, for some } n_0 \; f : \mathbb{N}_{\underline{\geq n_0}} \to \mathbb{R}, \; C, \text{ for some } n_0 \text{ s.t. } \|f(n)\| \leq C\underline{n}^3 \text{ if } \underline{n \geq n_1}, \text{ for some } n_1 \right\}$$

So

$$n^3 + (\log n) 10^{10^{10}} \in O(n^3)$$

$$-7n^3, \quad 3000 (\log n)^{10} \in O(n^3)$$

Here $\varepsilon \to 0$, in this context

$O(\varepsilon^2)$ or $\varepsilon^2 O(1)$

is the same class ~~of~~ of functions:

all $f = f(\varepsilon) : \mathbb{R}_{|\varepsilon| < \varepsilon_0} \longrightarrow \mathbb{R}, \mathbb{C}$

for some ε_0

s.t.

$$|f(\varepsilon)| \leq C \varepsilon^2$$

for some constant C,

all $|\varepsilon| \leq \varepsilon_1$

for some ε_1

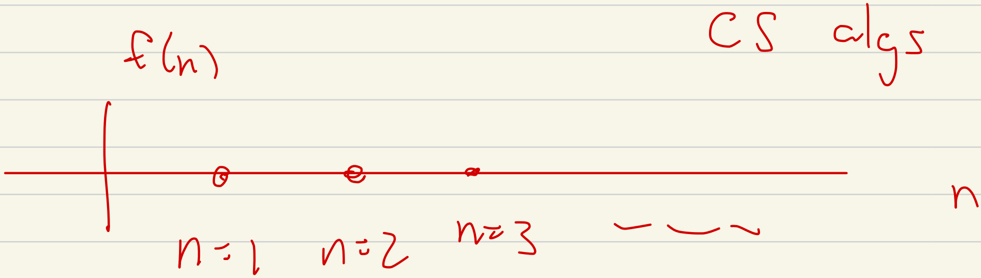
$$\frac{1}{1 + \underbrace{7\varepsilon^2 + \log(1/\varepsilon)\,\varepsilon^{10}}}$$

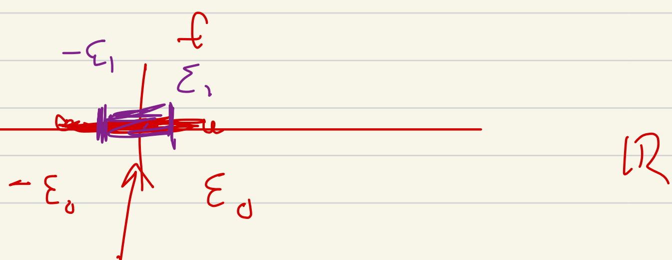
$\varepsilon \to 0$

$$= 1 - (\quad) + (\quad)^2 - (\quad)$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$$

CS algs





$f(\varepsilon)$ at $\varepsilon = 0$ may not be defined

$$\frac{1}{1-2\varepsilon^2} \qquad \varepsilon = 1/\sqrt{2}, \quad = \infty$$

$|\varepsilon|$ suff small

$$\frac{1}{1-2\varepsilon^2} = 1 + (2\varepsilon^2) + (2\varepsilon^2)^2 + \dots$$

$$= 1 + 2\varepsilon^2 + O(\varepsilon^4)$$

$$\frac{1}{1-x} = 1 + x + x^2 + \dots \qquad = 1 + O(\varepsilon^2)$$

$$= O(1)$$

$$\underbrace{\qquad\qquad = O\left(\frac{1}{\varepsilon^{20}}\right)}_{\text{"Galactic algorithms"}}$$

in theory efficient, in practice not ...

Udi Manber

$\Theta(n)$, $O(n^2)$

$OO(n^2)$ "uh-oh" of n^2

$$\left(10^{10^{10^{10^{20}}}} n + 3n^2\right)$$

$$= O(n^2)$$

$$= \underbrace{OO}_{\text{due to U. Manber}}(n^2)$$

Quasi-polynomial time:

works in time

FLOPs

$$\bigcup_{k \in \mathbb{N}} 2^{O((\log n)^k)}$$

$k=1$: $2^{O(\log n)} = n^{O(1)}$

= "polynomial in n"

$$R_A(\vec{u}+\varepsilon\vec{v}) =$$

$$\left(\left[A(\vec{u}+\varepsilon\vec{v})\right] - (\vec{u}+\varepsilon\vec{v}) \right) \frac{1}{\vec{u}\cdot\vec{u}}$$

$$\left(1 + O(\varepsilon^2)\right)$$

$$=$$

$$\left(\left[A(\vec{u}+\varepsilon\vec{v})\right] - (\vec{u}+\varepsilon\vec{v}) \right) \frac{1}{\vec{u}\cdot\vec{u}}$$

$$+ O(\varepsilon^2)$$

$$\left(A(\vec{u}+\varepsilon\vec{v})\right)\cdot(\vec{u}+\varepsilon\vec{v})$$

$$\left((A\vec{u})\cdot u\right)\varepsilon^0 \leftarrow \text{Constant}$$

$$+\ \varepsilon^1\left((A\vec{v})\cdot\vec{u} + (A\vec{u})\cdot\vec{v}\right)$$

$$+\ O(\varepsilon^2) \leftarrow O(\varepsilon^2)$$

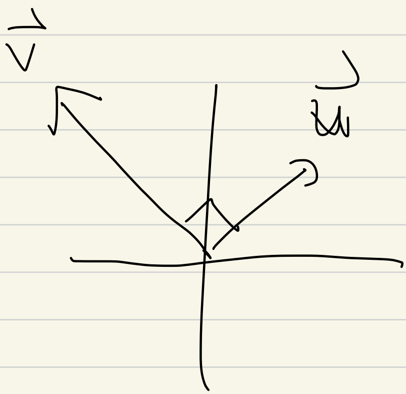
$$\left.\begin{array}{l}\underbrace{(A\vec{v})\cdot\vec{u}}_{(\vec{v}\cdot(A^T\vec{u}))}+(A\vec{u})\cdot\vec{v}\end{array}\right\} = 2\vec{v}\cdot(A\vec{u})$$

$$(A^T = A)$$

Hence, at maximum $\vec{u}$,

$$\vec{v} \cdot (A\vec{u}) = 0$$

whenever $\vec{v} \perp \vec{u}$



$$\Rightarrow \quad A\vec{u} = \lambda \vec{u} \text{ for some } \lambda \in \mathbb{R}$$

So $\vec{u}$ is an eigenvector,
whenever $\vec{u}$ maximizes
(or minimizes) $R_A(\vec{u})$!!

$$R_A(\vec{u}) = \frac{(A\vec{u}) \cdot \vec{u}}{\vec{u} \cdot \vec{u}}$$

$$= \frac{(\lambda \vec{u}) \cdot \vec{u}}{\vec{u} \cdot \vec{u}} = \lambda$$