CPSC 536F November 3, 2025

$A \in \mathbb{R}^{n \times n}$, $A^T = A$

Last time:

$$R_A(\vec{u}) = \frac{(A\vec{u}) \cdot (\vec{u})}{\vec{u} \cdot \vec{u}}$$

Rayleigh quotient

Rem: $$\left(f/g\right)' = \frac{1}{g^2}\left(f'g - fg'\right)$$

(calculus)

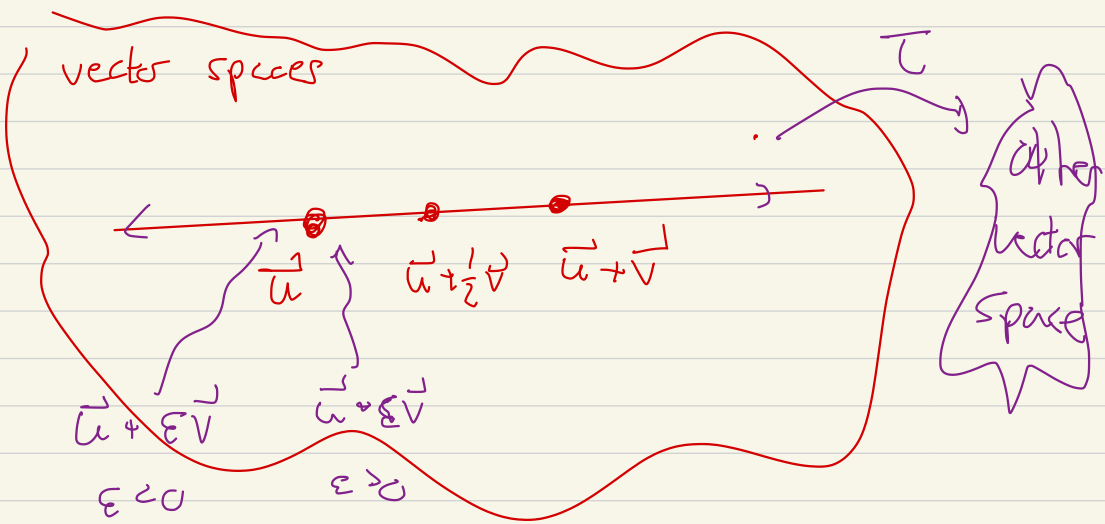
Compare $\alpha_0, \beta_0 \neq 0$

$$\frac{\alpha_0 + \varepsilon \alpha_1}{\beta_0 + \varepsilon \beta_1} = \alpha_0 \left(1 + \frac{\varepsilon \alpha_1}{\alpha_0}\right) \frac{1}{\beta_0} \left(\frac{1}{1 + \varepsilon \beta_1/\beta_0}\right)$$

$$= \frac{\alpha_0}{\beta_0}\left(1 + \varepsilon\left(\frac{\alpha_1}{\alpha_0} - \frac{\beta_1}{\beta_0}\right) + O(\varepsilon^2)\right)$$

Andrew Gleason (never got a PhD — super macho ...)

Claim: When working with ...
vector spaces, Banach spaces, etc
you only work with lines in these spaces: Variational methods

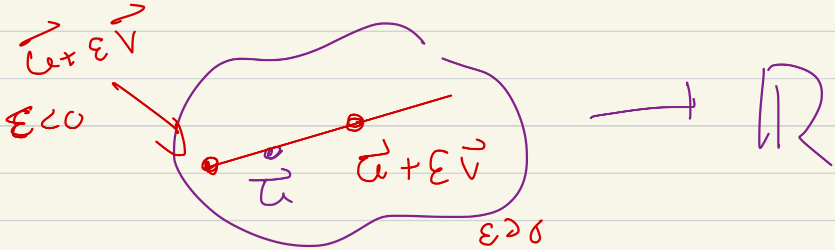


all you're doing $h(\varepsilon) = T(\vec{u}+\varepsilon\vec{v})$

"Variational method"

or global min / local min

If $\vec{u}$ is a maximum, ~~and~~ local maximum



OR saddle

then $\forall \vec{v}$, $h(\varepsilon) = I(\vec{u}+\varepsilon\vec{v})$,

$h'(0)$ must be zero

Say:

$$R_A(\vec{v}+\varepsilon\vec{u}) = \frac{f(\varepsilon)}{g(\varepsilon)}$$

where $f(\varepsilon) = \alpha_0 + \varepsilon\,\alpha_1 + O(\varepsilon^2)$

$g(\varepsilon) = \beta_0 + \varepsilon\,\beta_1 + O(\varepsilon^2)$

Assume $\frac{f(\varepsilon)}{g(\varepsilon)}$ has a local max at $\varepsilon = 0$ ..., $g(0) \neq 0$

$g(\varepsilon)$ differentiable near $\varepsilon = 0$

$\Rightarrow \left.\left(\frac{f(\varepsilon)}{g(\varepsilon)}\right)'\right|_{\varepsilon=0}$ must be 0

or ... $\nabla\left(\frac{f}{g}\right) = 0$

as appropriate

$R_A(\cdot) \quad \mathbb{R}^n \longrightarrow \mathbb{R}$

really $\mathbb{R}^n \setminus \{0\} \to \mathbb{R}$

only on S^{n-1}

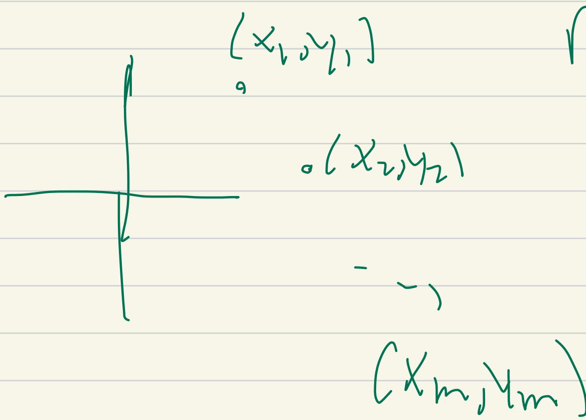
we should ~~talk~~ talk about

Solve $A\vec{u} = \vec{b}$

1

$$= \| A\vec{u} - \vec{b} \|^2$$

$$= (A\vec{u} - \vec{b}) \cdot (A\vec{u} - \vec{b})$$



Model

$$y = u_1 + u_2 x^2 + u_3 \sin(x)$$

Want smallest value of

$$y_1 = u_1 + u_2 x_1^2 + u_3 \sin(x_1)$$

$$y_2 = u_1 + u_2 x_2^2 + u_3 \sin(x_2)$$

$$\vdots$$

$$y_m = u_1 + u_2 x_m^2 + u_3 \sin(x_m)$$

$$\vec{b} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \text{ model } \approx \underbrace{\begin{bmatrix} 1 & x_1^2 & \sin(x_1) \\ 1 & x_2^2 & \sin(x_2) \\ \vdots & \vdots & \vdots \\ 1 & x_m^2 & \sin(x_m) \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{\vec{u}}$$

For the best solution $\vec{u}^*$

$$\mathcal{J}(\vec{u}^*) = \left\| A\vec{u}^* - \vec{b} \right\|_{L^2}^2$$

we look at

$$\mathcal{J}(\vec{u} + \varepsilon\vec{v})$$

$$= \left(A(\vec{u} + \varepsilon\vec{v}) - \vec{b} \right)$$

$$\cdot \left(A(\vec{u} + \varepsilon\vec{v}) - \vec{b} \right)$$

(Implicit in denominator $\Pr_A(\vec{u} + \varepsilon\vec{v})\ldots$)

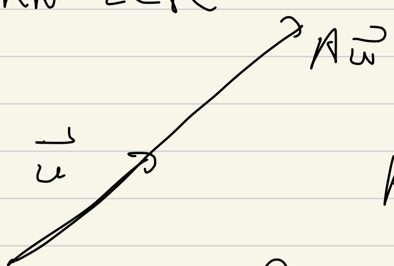
Our ultimate for eigenpairs of a symmetric matrix A, is to show that if

$$\vec{u} \text{ maximizes } R_A(\vec{u}) \stackrel{def}{=} \frac{(A\vec{u})\cdot\vec{u}}{\vec{u}\cdot\vec{u}}$$

then $\forall \vec{v}$ s.t. $\vec{v} \perp \vec{u}$

we have also $\vec{v} \perp A\vec{u}$

and in this case



$$A\vec{u} = \lambda \vec{u}$$

for some λ

(λ can be 0)

Lemma 1: If $A \in \mathbb{R}^{m \times n}$ matrix, $\vec{u}, \vec{v} \in \mathbb{R}^n$, $\vec{b} \in \mathbb{R}^m$,

$$f(\vec{u}) = \|A\vec{u} - \vec{b}\|_{L^2}^2$$

$$= (A\vec{u} - \vec{b}) \cdot (A\vec{u} - \vec{b}).$$

Then

$$f(\vec{u} + \varepsilon \vec{v}) =$$

$$\varepsilon^0 \, \text{Term}_0 + \varepsilon^1 \, \text{Term}_1 + \underbrace{O(\varepsilon^2)}_{\text{order } \varepsilon^2}$$

where $\text{Term}_1 = ???$

$$f(\vec{u} + \varepsilon \vec{v})$$

$$= \left(A(\vec{u} + \varepsilon \vec{v}) - \vec{b} \right) \cdot \left(A(\vec{u} + \varepsilon \vec{v}) + \vec{b} \right)$$

$$= \left((A\vec{u} - \vec{b}) + \varepsilon A(\vec{v}) \right) \cdot \left((A\vec{u} - \vec{b}) + \varepsilon A(\vec{v}) \right)$$

$\text{Term}_2 =$ whatever

$$\underline{\varepsilon\cdot\text{Term}_1} = \varepsilon(A\vec{v})\cdot(A\vec{u}-\vec{b})$$

$$+\ \varepsilon(A\vec{u}-\vec{b})\cdot(A\vec{v})$$

$$= \underline{2\varepsilon}\,(A\vec{v})\cdot(A\vec{u}-\vec{b})$$

$$= 2\varepsilon\left(\vec{v}\cdot \underbrace{A^T(A\vec{u}-\vec{b})}_{\text{"normal equations"}}\right)$$

Lemma:

$$\frac{\alpha_0 + \varepsilon \alpha_1 + o(\varepsilon)}{\beta_0 + \varepsilon \beta_1 + o(\varepsilon)} \quad \text{OR} \quad \frac{\alpha_0 + \varepsilon \alpha_1 + O(\varepsilon^2)}{\beta_0 + \varepsilon \beta_1 + O(\varepsilon^2)}$$

$$= \text{Term}_0 + \varepsilon \, \text{Term}_1 + \begin{cases} o(\varepsilon) \\ O(\varepsilon) \end{cases}$$

where

$$\text{Term}_1 = 0 \iff \frac{\alpha_1}{\alpha_0} = \frac{\beta_1}{\beta_0}$$