CPSC 536F Oct 31, 2025

Rayleigh quotients:

$$R_A(\vec{v}) = \frac{(A\vec{v})\cdot\vec{v}}{\vec{v}\cdot\vec{v}} = \frac{\vec{v}^T A^T \vec{v}}{\vec{v}^T \vec{v}}$$

$$\left(A = A^T \Longrightarrow \right) \quad = \frac{\vec{v}^T A \vec{v}}{\vec{v}^T \vec{v}}$$

$$A = \begin{bmatrix} 2 & & \\ & 5 & \\ & & 7 \end{bmatrix}, \quad R_A\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$$

$$= \frac{2x^2 + 5y^2 + 7z^2}{x^2 + y^2 + z^2}$$

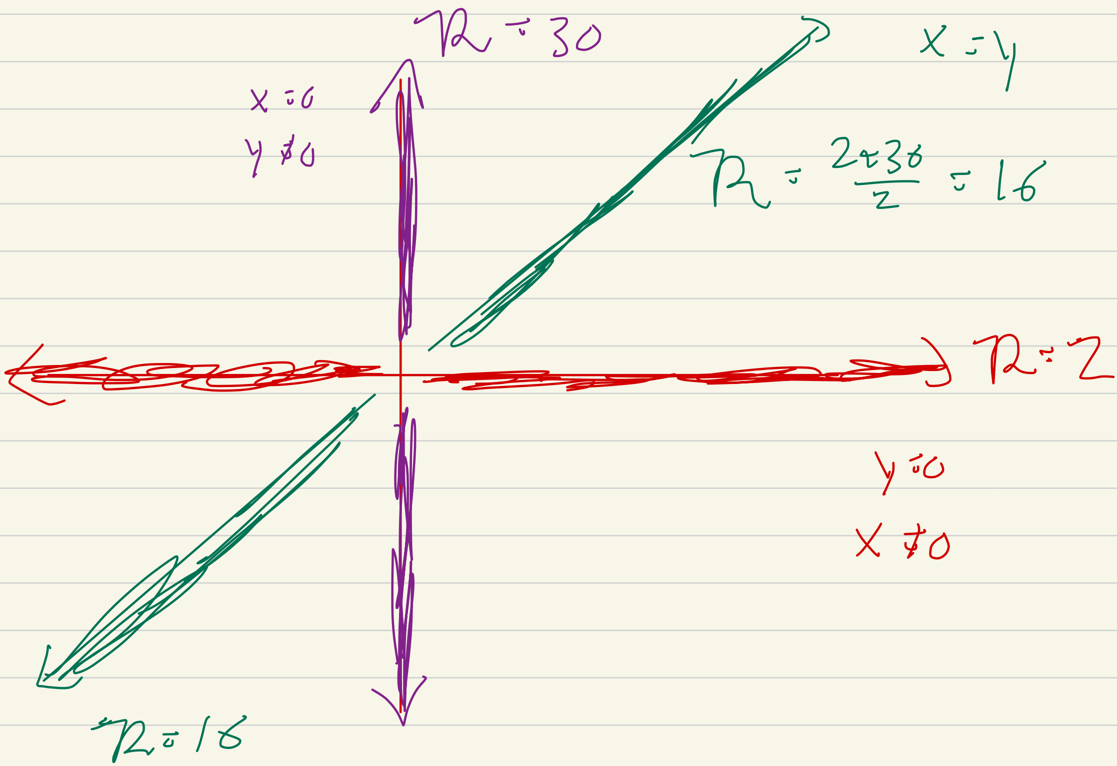
Claim:

$$\text{for } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{cases} \vec{e}_1 & \text{min} \\ \vec{e}_2 & \text{saddle} \\ \vec{e}_3 & \text{max} \end{cases}$$

$-1, 5,$

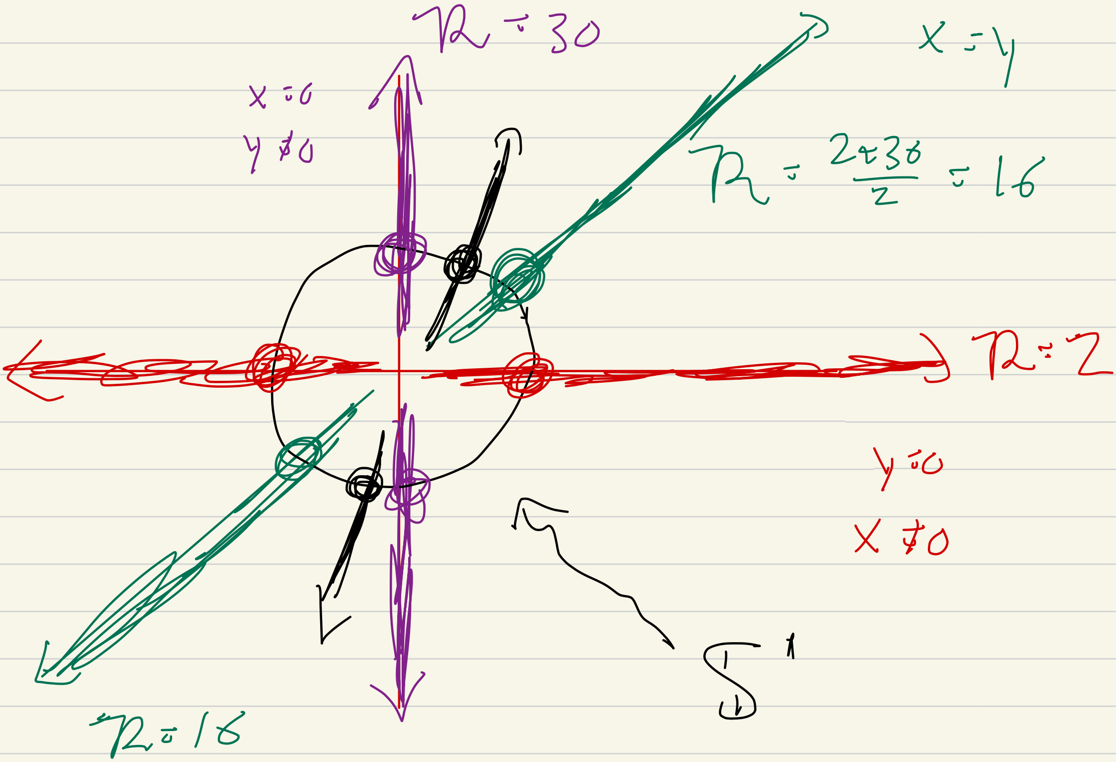
$\mathbb{R}^2$: $\begin{bmatrix} 2 & 0 \\ 0 & 30 \end{bmatrix}$

$$R_{\begin{bmatrix} 2 & 0 \\ 0 & 30 \end{bmatrix}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{2x^2 + 30y^2}{x^2+y^2}$$



-1.75

$\mathbb{R}^2$



Suffices to look at $x^2 + y^2 = 1$

Thm: Any continuous function on

$S^n = \{(x_1, \dots, x_{n+1}) \mid x_1^2 + x_2^2 + \dots + x_{n+1}^2 = 1\}$

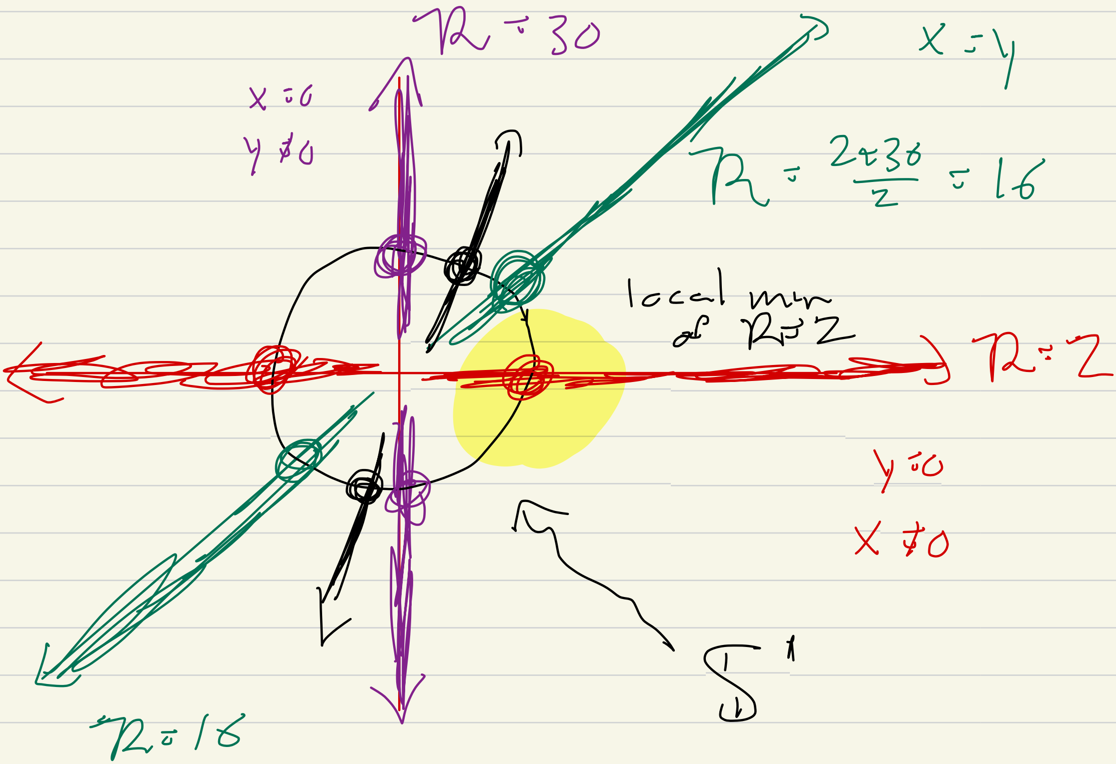
(or any compact subset of $\mathbb{R}^n$)

attains its maximum somewhere

(also -- minimum --).

E.g.

$\mathbb{R}^2$



N.B.

$$R_A(\alpha\vec{v}) = R_A(\vec{v}) \quad \forall \alpha \in \mathbb{R}_{\neq 0}$$

So really

R_A "lives" on sphere $x^2+y^2+z^2$

or $\mathbb{RP}^2$ or etc.

Thm: Fix symm $A \in \mathbb{R}^{n \times n}$. Consider $\vec{v}^*$ s.t.

$$R(\vec{v}^*) = \max R_A(\vec{v})$$

i.e.

notation:

$$\vec{v}^* = \arg\max_{\vec{v}} R_A(\vec{v})$$

(e.g. max likelihood estimators, etc.)

Then if

$$\lambda = R(\vec{v}^*) = \text{max value of } R_A$$

then

$$A\vec{v}^* = \lambda \vec{v}^*$$

and λ is the largest eigenvalue

Similarly max $\to$ min
largest $\to$ smallest

Rem: $R_{\begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}} = R_{\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}}$

If A is not symm

$$R_A = R_{(A+A^T)/2}$$

Proof:

$R_A(\vec{v})$ attains its max on

S^{n-1} at $\vec{v}^*$. Consider for

any $\vec{u} \in \mathbb{R}^n$ (write $\vec{v}$ for $\vec{v}^*$)

$$R_A(\vec{v} + \varepsilon \vec{u})$$

and, for simplicity, assume

$\vec{u} \perp \vec{v}$.

$$R_A(\vec{v} + \varepsilon \vec{u}) =$$

$$= \frac{A(\vec{v}+\varepsilon\vec{u}) \cdot (\vec{v}+\varepsilon\vec{u})}{(\vec{v}+\varepsilon\vec{u}) \cdot (\vec{v}+\varepsilon\vec{u})}$$

$$(\vec{v}+\varepsilon\vec{u}) \cdot (\vec{v}+\varepsilon\vec{u})$$

$$= \vec{v}\cdot\vec{v} + 2\varepsilon\,\vec{v}\cdot\vec{u} + O(\varepsilon^2)$$

order ε^2

really

$\varepsilon^2(\vec{u}\cdot\vec{u})$

$$= \vec{v}\cdot\vec{v} + O(\varepsilon^2) \quad , \quad \vec{v} \neq \vec{0}$$

$$\frac{1}{(\vec{v}+\varepsilon\vec{u}) \cdot (\vec{v}+\varepsilon\vec{u})} = \frac{1}{\vec{v}\cdot\vec{v} + O(\varepsilon^2)}$$

$$= \frac{1}{\vec{v}\cdot\vec{v}}\left(\frac{1}{1+O(\varepsilon^2)/\vec{v}\cdot\vec{v}}\right)$$

$\varepsilon \to 0$

$$= \frac{1}{\vec{v}\cdot\vec{v}}\left(1+\tilde{O}(\varepsilon^2)\right)$$

$$\left(\frac{1}{1+\varepsilon^2\alpha} \approx 1-\varepsilon^2\alpha\right)$$

$$A(\vec{v}+\varepsilon\vec{u})\cdot(\vec{v}+\varepsilon\vec{u})$$

$$= (A\vec{v})\cdot(\vec{v}) + \varepsilon\left(A\vec{u}\cdot\vec{v}+(A\vec{v})\cdot\vec{u}\right)$$

$$+\, O(\varepsilon^2)$$

$$= (A\vec{v} \cdot \vec{v}) + 2\varepsilon\left[(A\vec{u}) \cdot v\right] + O(\varepsilon^2)$$

since $A = A^T$

$$(A\vec{u}) \cdot \vec{v} \overset{\text{any } A}{=\!=} \vec{u} \cdot (A^T\vec{v}) \underset{A = A^T}{\overset{\text{symmetry}}{=}} \vec{u} \cdot (A\vec{v})$$

$$R_A(\vec{v} + \varepsilon\vec{u}) = \left(A(\vec{v} + \varepsilon\vec{u}) \cdot (\vec{v} + \varepsilon\vec{u})\right) \cdot \frac{1}{(\vec{v} + \varepsilon\vec{u}) \cdot (\vec{v} + \varepsilon\vec{u})}$$

$$= \varepsilon^0 \left(\text{Term}_0 \right)$$

$$+ \varepsilon^1 \left(2\left(A\vec{u} \cdot \vec{v}\right) / \vec{v} \cdot \vec{v} \right)$$

$$+ O(\varepsilon^2)$$

$$= \varepsilon^0 + \underbrace{\varepsilon^1 \text{ Proportional to } \left(A\vec{u} \cdot \vec{v}\right)}_{\text{must be } 0}$$

$$+ O(\varepsilon^2$$

$\Rightarrow$ If $\vec{u} \perp \vec{v}$

we must have $A\vec{u} \perp \vec{v}$

??

Lemma: $\forall A, \vec{v}, \vec{u}, \vec{b}$, fixed,

$\varepsilon \in \mathbb{R}$, $\varepsilon \to 0$

$$\| A(\vec{v} + \varepsilon \vec{u}) - \vec{b} \|^2$$

$$+ \varepsilon^0 \, \mathrm{Term}_0 + \varepsilon^1 \, \mathrm{Term}_1 + \varepsilon^2 \, \mathrm{Term}_2,$$

where

$$\mathrm{Term}_1 = \vec{u} \cdot \left(A^T A \vec{v} - A^T \vec{b} \right).$$

I.e.

$$\frac{d}{d\varepsilon} \| A(\vec{v} + \varepsilon \vec{u}) - \vec{b} \|^2 = \mathrm{Term}_1$$

Special case $A = I$, $\vec{b} = 0$