CPSC 536F Oct 29, 2025

– Finishing remarks on kernels, $k: \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ from Poisson equation $(-\Delta) w = f$ and

wow!

wow!

1. Gauge Field Theory (a rant or lecture by ChatGPT ...) and

$$\mathrm{ReLU}_0(x) \Big/ \{c_0 + x c_1\} = \left(\tfrac{1}{2}|x|\right) \Big/ \{c_0 + x c_1\}$$

Gaussian Kernel

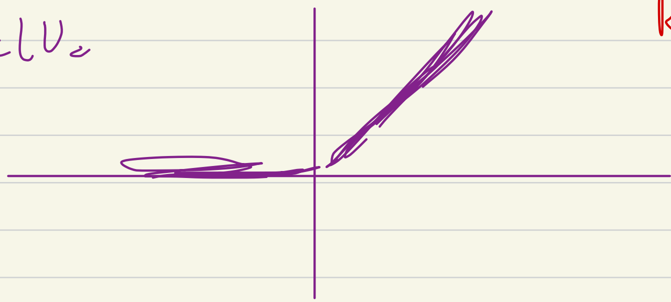
2. The heat equation

$w_t = \Delta w$, or $\underbrace{w_t = \mathcal{L}(w)}_{e^{\mathcal{L}t} w(0)}$

Easier

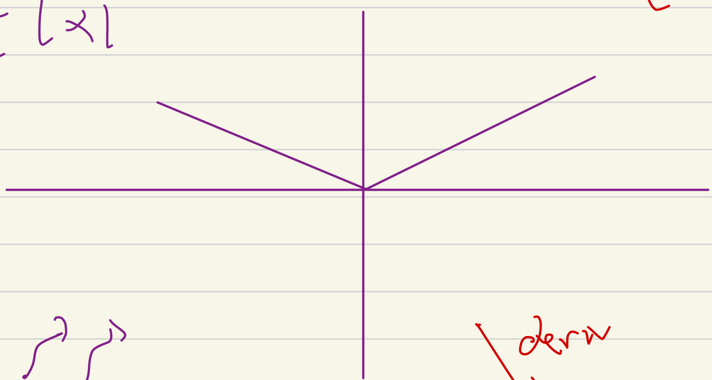
3. Fourier analysis on $(0,1)$, $(0,L)$

ReLU_0



$\mathrm{ReLU}_0'' = \delta_0$

$\frac{1}{2}|x|$



$\left(\frac{1}{2}|x|\right)'' = \delta_0$

symmetrix

deriv

$\left(\mathrm{Heaviside}(x) - 1/2\right)$

$$\mathrm{ReLU}_0'' - \left(\tfrac{1}{2}|x|\right)'' = \delta_0 - \delta_0 = 0$$

$$\Rightarrow$$

$$\left\{ \mathrm{ReLU}_0 + C_0 + C_1 x \right\}$$

$$= \left\{ \tfrac{1}{2}|x| + C_0 + C_1 x \right\}$$

Either:

$$\mathrm{ReLU}_0'' = \delta_0$$

OR

$$\left(\tfrac{1}{2}|x|\right)'' = \delta_0$$

ChatGPT: this is a "toy example" of Gauge Field Theory

To be continued if time permits...

and/or in the exercises

Today: Start (Section 4 of notes)

① Spectral theorem(s)

② Ridge regression

etc.

all via variational approach

((① can be proven via Schur decomp ...)

actually gives eigenvalues of Δ

on $\Omega = (0, L) \subseteq \mathbb{R}$

$\Omega \subset \mathbb{R}^n$ ...

Thm:

Let $A \in \mathbb{R}^{n \times n}$ be symmetric, i.e. $A^T = A$. We define $\forall\ \vec{v} \in \mathbb{R}^n \setminus \{0\}$:

$$\mathcal{R}_A(\vec{v}) = \frac{(A\vec{v}) \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$$ "Rayleigh quotient"

$$\underline{\underline{\text{matrix notation}}} \quad \frac{(A\vec{v})^T \vec{v}}{\vec{v}^T \vec{v}} .$$

We've assumed this

Then A has real eigenvalues

$$\lambda_n(A) \leq \quad \cdots \quad \leq \lambda_2(A) \leq \lambda_1(A)$$

and ON eigenbasis $A\vec{v}_i = \lambda_i \vec{v}_i$.

Pf: Consider -- $\mathcal{R}_A(\vec{v} + \varepsilon \vec{u}) \ldots$

Examples:

$$[1\ 7], \quad \begin{bmatrix} a & b \\ b & a \end{bmatrix} \quad \forall a, b \in \mathbb{R},$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

circulant matrices M s.t. $M^T = M$

Here we can write down eigenpairs! eigenfunctions, eigenvalues, vectors

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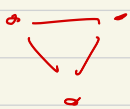
complete simple graph

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$



no self-loops

Any simple graph, especially d-regular graphs

Diagonal:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

eigenbasis

$\vec{e}_1, \vec{e}_2, \vec{e}_3$

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

λ, 2, 5, 7

Rem: Spectral thm $\Longrightarrow$ after an OrthoNormal change of basis, any symmetric matrix is $\begin{bmatrix} d_1 & & 0's \\ & d_2 & \\ 0's & & \ddots \end{bmatrix}$ diagonal matrix

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$$R_A(\vec{v}) = \frac{(A\vec{v}) \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$$

"Rayleigh quotient"

matrix notation

$$\frac{(A\vec{v})^T \vec{v}}{\vec{v}^T \vec{v}}$$

$$A = \begin{bmatrix} 2 & & \\ & 5 & \\ & & 7 \end{bmatrix}$$

convention: blank entries are 0

$$R_A\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$$

$\mathbb{R}^3$

$$= \frac{\left(A\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}}{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}}$$

assume $R_A(\vec{v})$ that $\vec{v} \neq \vec{0}$

$$= \frac{2x^2 + 5y^2 + 7z^2}{x^2 + y^2 + z^2}$$

Rem: x, y, z not all $= 0$,

$$2 \le \frac{2x^2+5y^2+7z^2}{x^2+y^2+z^2} \le 7$$

Pf: So

$$\frac{2x^2+5y^2+7z^2}{x^2+y^2+z^2} \le \frac{7x^2+7y^2+7z^2}{x^2+y^2+z^2}$$

$$= 7$$

When does

$$\frac{2x^2+5y^2+7z^2}{x^2+y^2+z^2} \le 7$$

hold with equality? Equivalently

$$\frac{2x^2+5y^2+7z^2}{x^2+y^2+z^2} \le \frac{7x^2+7y^2+7z^2}{x^2+y^2+z^2}$$

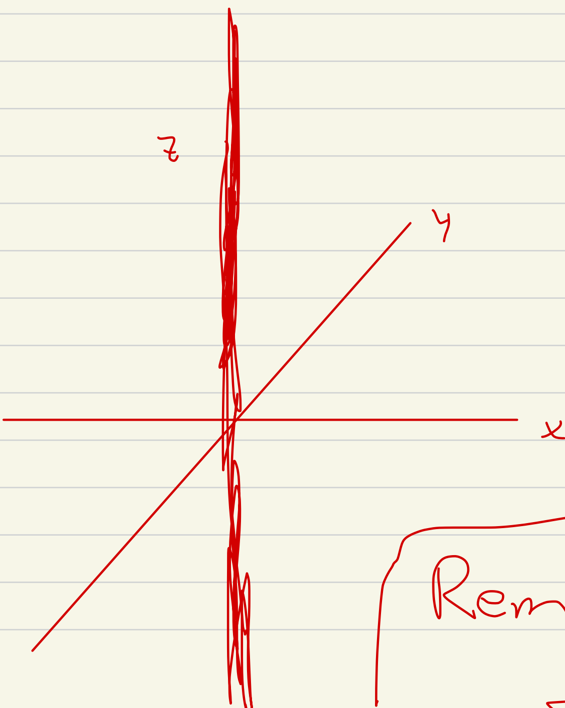
holds with equality?

$$\left.\begin{aligned} x \neq 0 &\quad 2x^2 < 7x^2 \\ y \neq 0 &\quad 5y^2 < 7y^2 \end{aligned}\right\} \quad \text{So equality} \Rightarrow x, y = 0$$

If $x, y = 0$, $z \neq 0$

So

$$\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = z \cdot \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\vec{e}_3}$$



Rem: $R_A(\alpha \vec{v}) = R_A(\vec{v})$ for all $\alpha \in \mathbb{R}$

Other eigenvectors look like saddles of some kind