CPSC 536 Oct 22, 2025

- Next week:

Section 4:

- Variational proof of the Spectral theorem for finite symmetric matrices

- Variational analysis of regression and ridge regression

- Connection to kernels

- SVD, PCA

- This week: say as much as possible about:
  - Generalized Functions
  - Solving $w'' = \delta_y(o)$ on $(0, L)$ with Dirichlet conditions
  - Why Green's function on $(0, L)$ is
    - Used to solve Poisson equation $\pm \Delta w = f$
    - Is positive definite (as a kernel function)
  - Heat equation and Gaussians

- Some material that we don't get to will appear as exercises

- I may add some questions to ask ChatGPT / Gemini / etc. when you are ready to learn more... (not homework to be submitted...)

ChatGPT / Gemini do a great job if you want to know! the common notation in Avner Friedman...

Joel's notes (for Joel...)

$$\left.\begin{array}{l}\boxed{\mathcal{D}(\mathbb{R})} \\ \boxed{\mathcal{D}(\Omega)}\end{array}\right\} \text{ implies } \begin{array}{l} C_c^\infty(\mathbb{R}) \\ C_c^\infty(\Omega)\end{array}$$

Ω open, e.g. $(0,L) \subset \mathbb{R}$

$\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$ means:

(1) $\operatorname{supp}(\varphi_n) \subset K$ for some compact $K \subset \Omega$

(2) $\forall\, k \in \mathbb{N}$, $\max\left|\varphi_n^{(k)} - \varphi^{(k)}\right| \to 0$

$\mathcal{D}'(\Omega) \overset{\text{def}}{=}$ <u>continuous</u> linear functionals on $\mathcal{D}(\Omega)$; i.e.

$\{T \mid \forall\, \varphi_n \to \varphi,\ \langle T, \varphi_n \rangle \to \langle T, \varphi \rangle$

where $\langle \, , \rangle$ is the duality pairing

$$\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega) \to \mathbb{R}$$

Most intuitive properties of ODEs hold for distributions: e.g.

solving

$$w' = f \qquad \text{on } (0,L) \text{ or } \mathbb{R}:$$

two solutions

$$w_1' = f, \quad w_2' = f$$

have

$$w_1' - w_2' = C \qquad \text{constant}$$

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functions: solve $w' = f$, $\mathbb{R}$,

sol:

$$w(x) = \int f(x)\,dx + C$$

$w_1' = f = w_2'$ regular functions

then

$$w_1(x) - w_2(x) = C.$$

Exercise: Same is true for generalized functions

This should doable w/o knowing

Generalized functions $\overset{\text{def}}{=}$

$\boxed{\text{continuous}}$ linear maps

$$C_c^\infty(\mathbb{R}) = \mathcal{D}(\mathbb{R}) \to \mathbb{R}$$

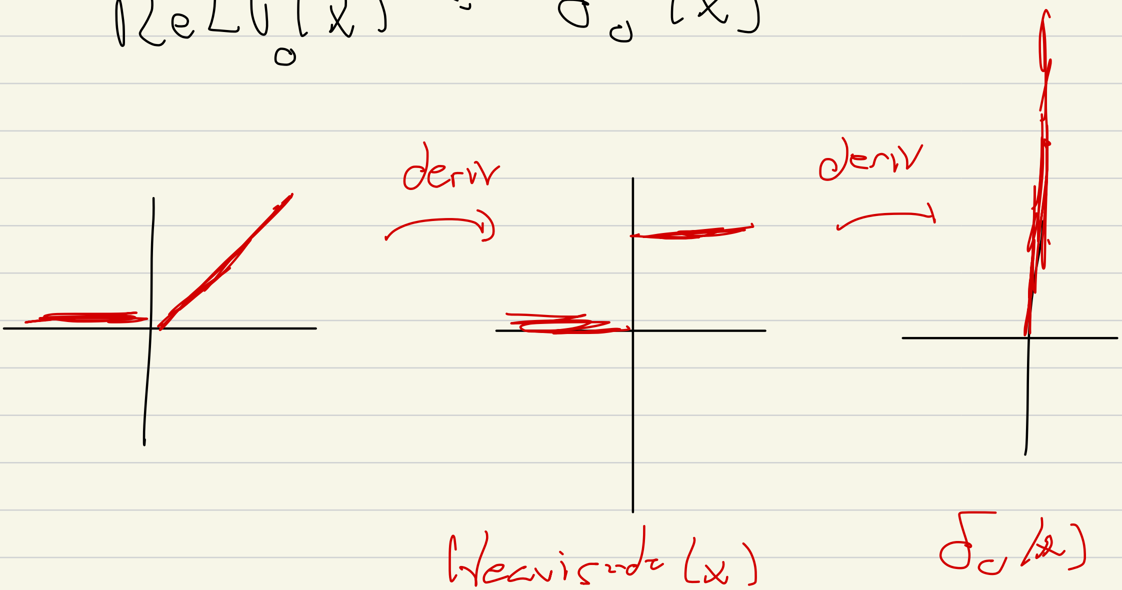
w/o knowing ...

Back to "naive" calculations:

$$\mathrm{ReLU}(x) = \mathrm{ReLU}_0(x)$$

has

$$\mathrm{ReLU}_0''(x) = \delta_0(x)$$



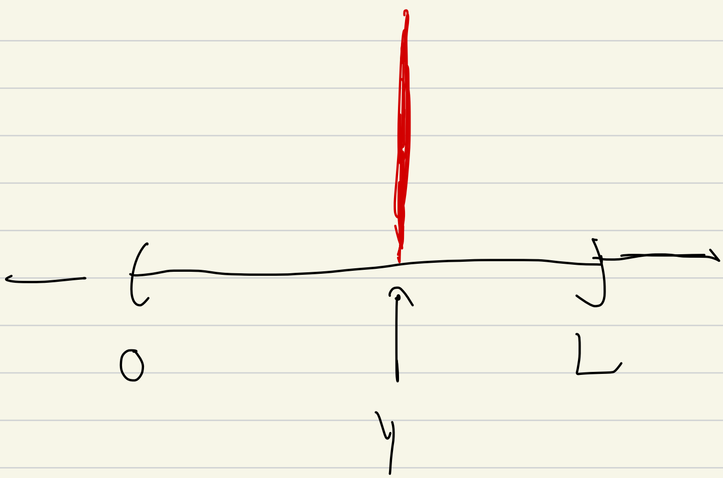
Solve, fix $y \in (0, L)$

$$-w''(x) = \delta_y(x) \qquad x \in (0, L)$$

Dirichlet conds

$$w(0) = 0, \quad w(L) = 0$$

Here $\delta_y(x) = \delta_0(x - y)$



$$\left(\mathrm{ReLU}_y(x)\right)'' = \delta_y(x)$$

$$\mathrm{ReLU}_y(x) = \mathrm{ReLU}(x-y)$$

$$= \mathrm{ReLU}_0(x-y)$$

$\downarrow$

General solution to $-w'' = \underbrace{\delta_y}_{\text{any}}(x)$

$C_0 + x C_1 +$ particular solution

here

$C_0 + x C_1 + \mathrm{ReLU}_y(x)$

Now:

$$C_0 + x\,C_1 + \mathrm{ReLU}_y(x) = \omega(x)$$

to satisfy

$$\omega(0) = \omega(L) = 0$$

So

(1) $\omega(0) = C_0 \implies C_0 = 0$

(2) $\omega(x) = x\,C_1 + \mathrm{ReLU}_y(x)$

$\omega(L) = 0$

$$0 = L\,C_1 + \mathrm{ReLU}_y(L)$$

$$w(x) = x\,C_1 + \mathrm{ReLU}_y(x)$$

$$\uparrow$$

$$0 = L\,C_1 + \mathrm{ReLU}_y(L)$$

$$-L\,C_1 = \mathrm{ReLU}_y(L)$$

$$(0 \leq y \leq L) \qquad = L - y$$

$$C_1 = \frac{L - y}{-L} = \frac{y - L}{L} = \frac{y}{L} - 1$$

Call this

$$\boxed{\text{Green}(y, x)}$$

$$= \text{ReLU}_y(x) + c_1 x$$

$$= \boxed{\text{ReLU}_y(x) + \left(\frac{y}{L} - 1\right) x.}$$

Solve $w_1'' = f$, $w_2'' = f$

$$(w_1 - w_2)'' = 0$$

$$\Rightarrow \quad w_1 - w_2 = C_0 + x C_1$$

Claim: $G(y,x) = G(x,y)$,

i.e. $\mathcal{X} = [0,L]$ is a

symmetric kernel function

$$\mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}$$

$$(x\ ,\ y) \longmapsto G(x,y)$$

(we are extending from $\mathcal{X} = (0,L)$
to $\mathcal{X} = [0,L]$ if you want)

Exercise: Not too painful.

Reason: $\int (-w'')(v) = \int w(-v'')$

modulo boundary terms of twice integrating by parts,

(Exercise)

(because of Dirichlet conditions)

i.e. $w \longmapsto w''$ is self-adjoint

by contrast

$$\int (w')v = \int -(w v')$$

is $G(x,y) = -G(y,x)$

Claim 2: $G(x,y)$ is positive definite,

i.e. $x_1, \dots, x_m \in (0,L)$ or $[0,L]$

any $\alpha_1, \dots, \alpha_m \in \mathbb{R}$

then

$$\sum_{i,j} \alpha_i \alpha_j G(x_i, x_j) > 0$$

unless $\alpha_1 = \alpha_2 = \ldots = \alpha_m = 0$

i.e.

$$\sum_{i,j} \alpha_i \alpha_j \left(\mathrm{ReLU}_{x_i}(x_j) + \text{messy linear term} \right) > 0$$

Make it simpler:

$0 \le x_1 < x_2 < \ldots < x_m \le L$

ReLU simplifies

Friday!

Gaussians $\longleftrightarrow$ Heat Eq

Finish Generalized Functions

Start on variational proof of spec theorem

Class ends

A.2.(e)

$$r_1^2 + r_2^2 + \cancel{r^2}$$

but

$$r_1^2 + r_2^2 + r^2/2$$

Jiang