CPSC 536F Oct 20, 2025

Bad news: I'm still a bit behind

Good news: I have managed to come up with a few nice exercises on generalized function, to explain what we need from!

Generalized Functions and PDE
A. Friedman

Greats new: You and I will study non-parametric stats
(Thanks to a conversation with Ali Lazrak)

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Ultimate goals:

1. Positive definite kernel functions

$\mathcal{X} \times \mathcal{X} \to \mathbb{R}$ s.t. $\mathcal{X} = \mathbb{R}, \mathbb{R}^2, \ldots$

2. Explain "Gaussian kernel" in terms of the heat equation

3. "Toolbox" to build kernel functions + understanding them

$\rightsquigarrow$ Non-parametric methods (uses kernel functions)

Last time: weak solution to $w' = f$

is pair

$$(w, f) = (\mathrm{ReLU}(x), \mathrm{Heaviside}(x))$$

Technically we need:

For test functions $\Psi = C_0^\infty(\mathbb{R}) = C_c^\infty(\mathbb{R})$

0 or c

compact

we have

1 $(\mathrm{Heaviside}(x), \delta_0(x)) = (w, f)$

also satisfies

$$w' = f$$

what does δ_0 really mean -- ?

Ans: $\delta_0(x) : \underbrace{C_0^\infty(\mathbb{R})}_{\Psi} \to \mathbb{R}$

$\Psi \to \mathbb{R}$

1' generalized function

2 If $f, g : \mathbb{R} \to \mathbb{R}$ are continuous,

then $\underbrace{f = g}_{\text{as functions}} \iff \underbrace{f = g}_{\text{as generalized functions}}$

Exercises

[not really sure $\Psi = C_0^\infty(\mathbb{R})$ is "best" choice ...

3 Exercise: if $(g_1, g_2), (g_2, g_3)$ are (w, f) s.t. $w' = f$, is $g_1 \text{"} = g_3$??

I.e. is

(g_1, g_2) a weak solution

(w, f) to

$$w' = f.$$

Last time ReLU$'$ = Heaviside
(weakly)

Now:

$$\text{Heaviside}' = \delta_0$$

what is this

Weak solutions:

$$w' = f \quad ??$$

$g \in \Phi$ test functions, here

$$\Phi = C_c^\infty(\mathbb{R})$$

f replaced by

$$g \longmapsto \int_{-\infty}^{\infty} f(x)g(x)dx$$

i.e. a map

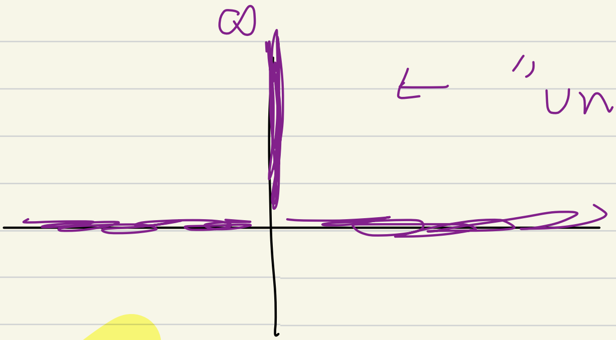
$$\underbrace{\qquad\qquad}$$

$$\Phi \longrightarrow \mathbb{R}$$

$C_c^\infty(\mathbb{R})$

Intuitively, if $g \in C_0^\infty(\mathbb{R})$

$\delta_0 = \delta_0(x)$ the Dirac delta at $x=0$



← "unit impulse"

what should be

This is the key

$$\int_{x=-\infty}^{x=\infty} g(x)\,\delta_0(x)\,dx = g(0)$$

(!!!)

Recall $C^\infty(\mathbb{R})$ means

infinitely differentiable $\mathbb{R} \to \mathbb{R}$

$C^0(\mathbb{R})$, $C^1(\mathbb{R})$ ---

$C^k(\mathbb{R})$ = k-times differentiable with k-th derivative is continuous

$C^0(\mathbb{R})$ $\quad$ $C^0((0,L))$ -->

continuous

$C^k_c(\mathbb{R})$ or $C^k_0(\mathbb{R})$

means having compact support

<u>Def 1:</u> A <u>generalized function</u>

(for $C_c^\infty(\mathbb{R})$) is a linear

[might add more] map

$$C_c^\infty(\mathbb{R}) \longrightarrow \mathbb{R}$$

[f continuous

$$g \longmapsto \int_{\mathbb{R}} f(x)g(x)\,dx$$
]

[then

$$(\alpha_1 g_1 + \alpha_2 g_2) \mapsto \alpha_1 \int f g_1 + \alpha_2 \int f g_2$$]

e.g.,

f continuous

$$g \longmapsto \int_{\mathbb{R}} f(x)\, g(x)\, dx$$

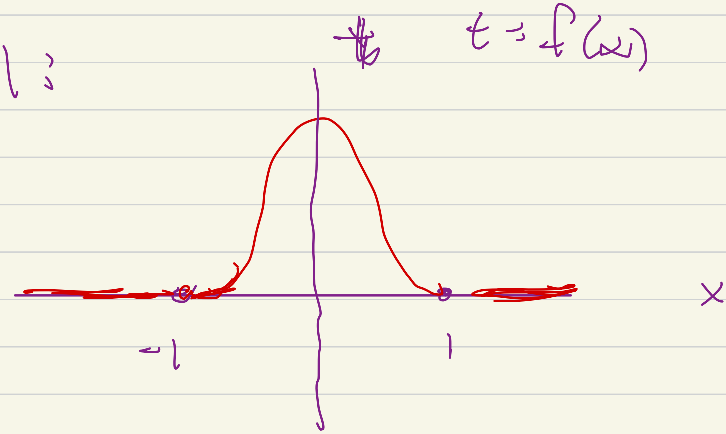
e.g.,

δ_0

$$g \longmapsto g(0)$$

... What could possibly go wrong?

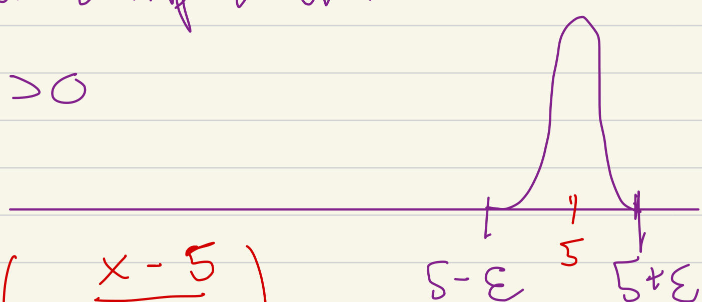
Well:



$$f(x) = \begin{cases} 0 & |x| \geq 1 \\ e^{\frac{-1}{1-x^2}} & x < 1 \end{cases}$$

Also similar bump function:

for any $\varepsilon > 0$



$$f\left(\frac{x-5}{\varepsilon} \right)$$

Really: "better bump"

$$g(x) = C\, f(x)$$

where $C > 0$ is chosen s.t.

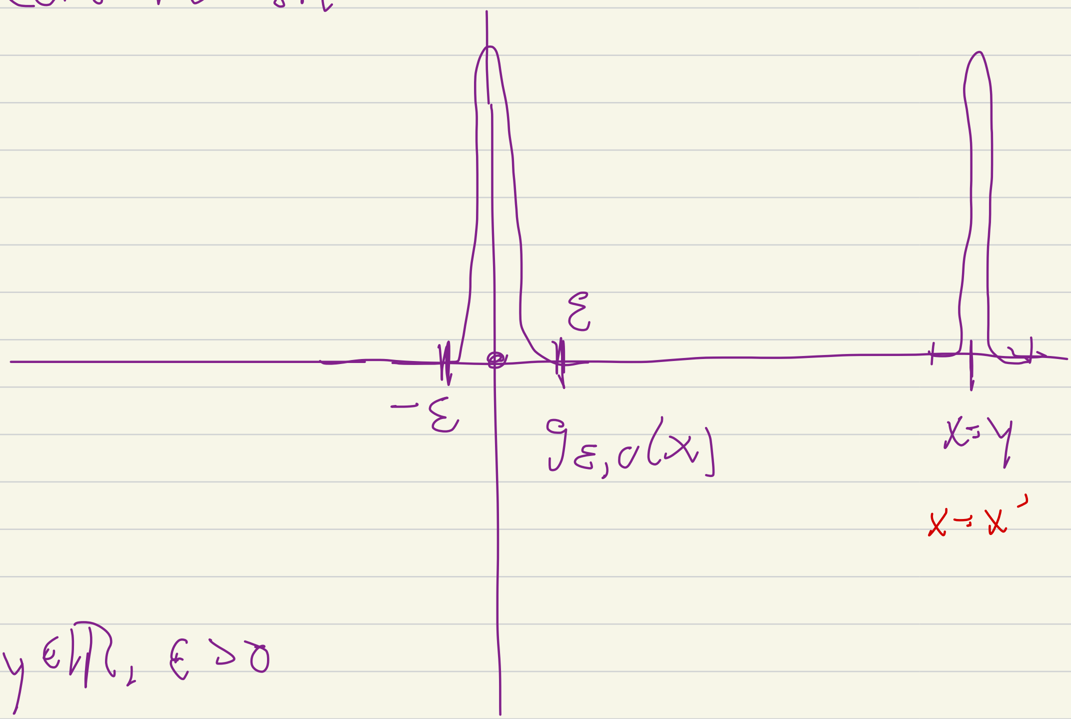
$$\int_{-\infty}^{\infty} g(x)\, dx = 1$$

Similarly:

$$g_\varepsilon(x) = C_\varepsilon\, f\left(\frac{x-5}{\varepsilon}\right)$$

s.t. $\int_{-\infty}^{\infty} g_\varepsilon(x)\, dx = 1$

i.e. $g_\varepsilon(x)$ is the shift of f times the correct constant s.t.



$\forall y \in \mathbb{R},\ \varepsilon > 0$

$$g_{\varepsilon,y}(x) = g_\varepsilon(x-y)$$

OR, in ML $g_{\varepsilon,x'}(x) = g_\varepsilon(x-x')$

$k(x, y)$ in math

$k(x, x')$ in ML

Next time

① What possibly go wrong by considering all linear maps $C_0^\infty(\mathbb{R}) \to \mathbb{R}$

② if $f = g$ as continuous functions, if $f = g$ as generalized functions

Class Ends

Each

Example: C_0^∞ larger/smaller

C^{anal} 〃 〃