CPSC 536F Oct 17, 2025

① Show that

$$\mathrm{ReLU}'(x) = \mathrm{Heaviside}(x) \quad ???$$

in the "weak" sense.

amazingly powerful technique

② What about

$$\mathrm{Heaviside}'(x) = \delta(x) \quad ??$$

(all centred at 0, i.e.

$$\delta(x) = \delta_0(x)$$

$$\mathrm{Heaviside}(x) = \mathrm{Heaviside}_0(x)$$

$$\mathrm{ReLU}(x) = \mathrm{ReLU}_0(x)$$
)

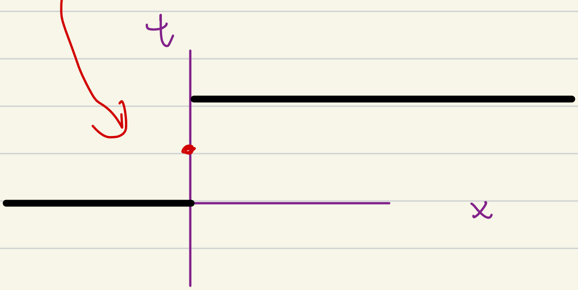
and we will write $y \in \mathbb{R}$

$$\delta_y(x) \overset{\text{def}}{=} \delta_0(x-y)$$

$$\text{Heaviside}_y(x) \overset{\text{def}}{=} \text{Heaviside}_0(x)$$

$$\text{ReLU}_y(x) = \text{ReLu}_0(x-y)$$

[Maybe $\text{Heaviside}_0(0) = 1/2$]



$$t = \text{Heaviside}(x)$$
$$= \text{Heaveside}_0(x)$$

centred at 0

$$\text{Heaviside}_0(0) = ???$$

(2') What is $\delta(x) = \delta_0(x)$ anyway?

Answer: It is a <u>generalized function</u> meaning, a linear map $\Psi \leftarrow$ "test functions"

$$\Psi : \overbrace{C_0^\infty(\mathbb{R})} \longrightarrow \mathbb{R}$$

(sometimes subject to certain conditions, e.g.

$f \in C^1 \iff$ etc.

Think of:

$$C_0^\infty(\mathbb{R}) \longrightarrow \mathbb{R}$$

$$\underset{\substack{\uparrow \\ \text{function}}}{\Phi_f} : \quad C_0^\infty(\mathbb{R}) \longrightarrow \mathbb{R}$$

$$g \longmapsto \int f(x)\,g(x)\,dx$$

③ Is

$$\text{ReLU}_0''(x) \overset{???}{=} \delta_0(x)$$

$$w''(x) = \begin{cases} \text{anything given...} \\ \text{could even be} \\ \delta_0(x) \end{cases}$$

Trick / Method (add M. Hirsch)

$w \in C_0^\infty(\mathbb{R})$

$$\int w''(x)\, w(x)\, dx = -\underbrace{\overbrace{\int \left(w'(x)\right)^2 dx}}_{\text{looks like it has to be pos (semi) definite}}$$

$\Rightarrow$ $w \mapsto w''$ is negative semidefinite

Exercises to be written: We'll prove some of these ideas

2.5

If Ψ test functions are taken

to be $\Psi = \{0\}$

the function

$$\Psi_f : g \longmapsto \underbrace{\int g(x) f(x) dx}_{\text{should be } 0}$$

$f \uparrow 0$

then any function s.t. blah blah ...

should equal any other function ...

So if

$\Psi = C_c^\infty(\mathbb{R})$ need to show there

are "enough functions"

To prove (1): $w'(x) = f(x)$

So $\boxed{\int_{\mathbb{R}} g(x)\, \underbrace{w'(x)}\, dx} = \int_{\mathbb{R}} g(x) f(x)\, dx$

formally

just need Riemann sum (integral)

$$-\int_{\mathbb{R}} g'(x)\, w(x)\, dx = \int_{\mathbb{R}} g(x) f(x)\, dx$$

Since $g \in C_0^\infty(\mathbb{R})$

$g' \in C_0^\infty(\mathbb{R})$, $w(x), f(x) \in \overbrace{C^0(\mathbb{R})}^{\text{continuous}}$

$$\mathrm{ReLU}'(x) = \mathrm{Heaviside}(x)$$

in weak sense ??

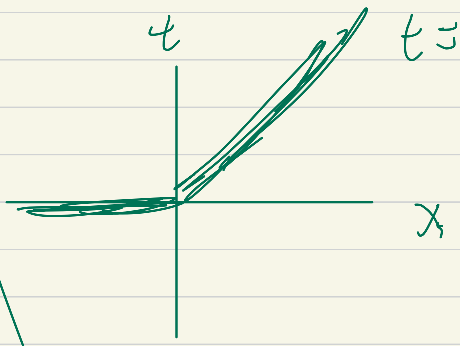
Is $\forall g \in C_c^\infty(\mathbb{R})$:

$$-\int_{\mathbb{R}} g'(x)\,\mathrm{ReLU}(x)\,dx$$

$$= \int_{\mathbb{R}} g(x)\,\mathrm{Heaviside}(x)\,dx$$

???

$$= \int_{\mathbb{R}} g'(x)\, \mathrm{ReLU}(x)\, dx$$



$$= \int_{x=0}^{x=\infty} g'(x)\, x \, dx$$

$$= g(x)\,x \Big|_{x=0}^{\infty} + \int_{x=0}^{x=\infty} g(x) \left(\frac{dx}{dx}\right) dx$$

$$= \int_{x=0}^{x=\infty} g(x)\, dx = \text{whatever}$$

$$\int_{x=-\infty}^{x=\infty} g(x)\, \text{Heaviside}(x)\, dx$$

$$\overset{???}{=} \int_{x=-\infty}^{x=\infty} g(x) \begin{Bmatrix} 1 & x>0 \\ 0 & x<0 \end{Bmatrix} dx$$

$$= \int_{x=0}^{x=\infty} g(x)\, dx$$