CPSC 536F   Oct 15, 2025

Bad news / good news:

Bad news: I've only begun to give you feedback on September homework

first in first out

Good news: I'm trying for FIFO

Bad news: If you want to truly understand generalized functions you need to read (cover to cover):

(1) Partial Differential Equations

\$30 ⇝ - Avner Friedman

(2) Generalized Functions and P.D.E.

\#84 ⇝ - Avner Friedman

(Now first 10-20 pages)

Good news!

I'll summarize what we need to understand how to build positive definite kernels on $\mathcal{X}$,

$$|\mathcal{X}| = |\mathbb{R}| = \aleph_1 = 2^{\aleph_0} = 2^{|\mathbb{N}|}$$

Great news!

Heat equation and Gaussian kernels

Counting, cardinal number

$0 < 1 < 2 < 3 < \ldots,$ $\aleph_0$,

$\uparrow$ $\quad$ $\uparrow$

$\emptyset$ $\quad$ first infinite cardinal number

$$\aleph_0 < \text{Power}(\aleph_0)$$

set $\simeq \mathbb{N} \simeq \mathbb{Q} \simeq$ countably infinite

sets $\simeq \text{Power}(\mathbb{N}) \simeq \mathbb{R}$

Question: is there a cardinal number between $\aleph_0$ and $\aleph_1$

$$\aleph_0 < \ ? \ < \aleph_1 \qquad ??$$

Solved in 1960's

Paul Cohen
~~S~~Stanford U

independent of ZFC ??

~ ZF

ETC

Last time: We were solving the Poisson equation

$$w''(x) = f(x) \qquad x \in [0, L],$$

$$w(0) = w(L) = 0. \quad \leftarrow \text{Dirichlet boundary conditions}$$

~~We~~ I claimed the solution is

$$w(x) = \int_0^L \underbrace{f(t)\, G(t, x)}_{\leftarrow \text{ image}}\, dt$$

$$e^{e^{(\sin 3x + 2)}}$$

where

$$\underbrace{G(t, x)}_{\text{Green's function}} = \mathrm{ReLu}(x - t) + \text{simple stuff}$$

[Recall that Green's function predates modern ML by $\approx 2 \cdot 10^2$ years]

So we should have

$$\left(\frac{d}{dx}\right)^2 w(x) = \int_0^L f(t) \left(\frac{\partial}{\partial x}\right)^2 G(t,x) \, dt$$

Here

$$\left(\frac{d}{dx}\right)^2 \mathrm{ReLU}_0(x)$$

should make sense. But clearly even

$$\frac{d}{dx} \mathrm{ReLU}_0(x) = \frac{d}{dx} \mathrm{ReLU}(x)$$

makes no sense ...

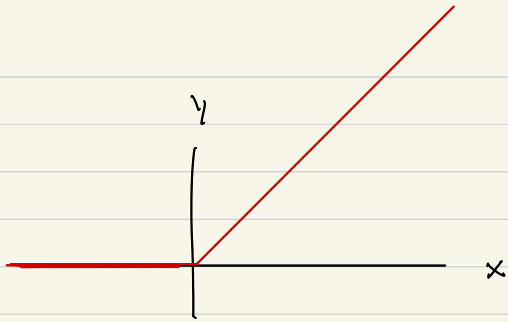
... Indeed,

Translate

$\mathrm{ReLU}_y(x)$

$= \mathrm{ReLU}(x-y)$

ReLU(x)



$$\frac{d}{dx}\,\mathrm{ReLU}(x) = \begin{cases} \mathrm{Heaviside}(x) & x \neq 0 \\ ??? & x = 0 \end{cases}$$

Solution: Consider "weak" solutions to the ODE

$$w'(x) = f(x) \qquad \forall x \in \mathbb{R}$$

$$\left(w' = f(x, w) \right)$$

solution:

$$w(x) = \int f(x)\,dx + C$$

weak solution: a very powerful method in PDE's:

To get a more general context:

let Ψ be any subset of functions $\mathbb{R} \to \mathbb{R}$. A weak solution to

$$w'(x) = f(x), \quad x \in \mathbb{R}$$

for w, f where w is continuos, but not literally differentiable:

$\forall g \in \Psi$ $\quad \Psi$ is called a

class
set of test functions*

$$\int_{\mathbb{R}} g(x)\, w'(x)\,dx$$

$$= \int_{\mathbb{R}} g(x)\, f(x)\,dx,$$

... but ... $w'(x)$ still makes no sense ... ☹

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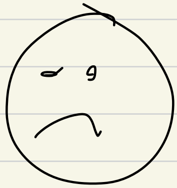
*in A. Friedman, Generalized Functions and PDE, Ψ is written Φ

Note:

If $\Psi = \emptyset$



Ψ = vector space

-- Schwatz ≈ 1940's --

Take

$$\Psi = C_0^\infty(\mathbb{R})$$

Schwartz famous for another class

why -- point:

if $h_1(x) = h_2(x) \quad \forall x \in \mathbb{R}$

$$\Leftrightarrow \int g h_1 = \int g h_2 \qquad \forall g \in \Psi$$

However, if $g, h \in C_0^\infty(\mathbb{R})$, smooth (infinitely differentiable) and of compact support (i.e. $g(x) = h(x) = 0$ for $|x|$ large)

note that

$$\boxed{\int_{\mathbb{R}} g(x)h'(x)\,dx = \int -g'(x)h(x)\,dx \,. \qquad \text{(eq. 1)}}$$

Hence $gh' \rightsquigarrow (-g')h$,

and now we require:

NOT

$$\forall x \in \mathbb{R}, \quad w'(x) = f(x)$$

RATHER:

[thinking, (eq. 1)]

$$\int g w' = \int g f$$

$$\|$$

$$-\int w g'$$

— 9a —

doesn't make sense ..

$$\int g w' = \int g f$$

$$= \int w(-g')$$

We require: $\forall g \in C_0^\infty(\mathbb{R})$

$$\int w(x)(-g'(x))\,dx$$

$$= \int g(x) f(x)\,dx$$

Everything makes sense, w, f only continuous

[Now if $\Psi \in C_0^\infty(\mathbb{R})$

(or something smaller or larger)]

Claim:

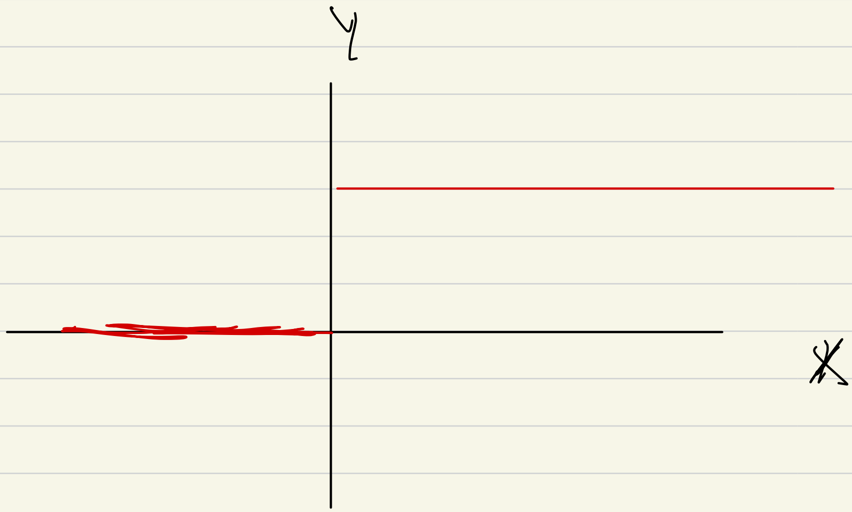
$$\left(\mathrm{ReLU}(x)\right)' = \mathrm{Heaviside}(x)$$

in the weak sense, where

$$\mathrm{Heaviside}(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \\ ?? & x = 0 \end{cases}$$

don't care

Heaviside(x)



$x = 0$???

Thm / Prop / Exercise (Exercise not in 2025)

$$\frac{d}{dx} \mathrm{ReLU}(x) = \mathrm{Heaviside}(x)$$

as an ODE solution in the <u>weak</u> sense

Proof: Blah blah blah (Friday)

Friday:

(1)

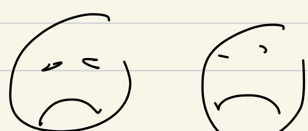
$$\frac{d}{dx}\,\text{Heaviside}(x) = \delta_0(x)$$

as a ==weak== ODE solution

Heaviside

$$\delta_0(x) = \delta(x) = \text{"unit impulse at } x=0\text{"}$$

$$= \begin{cases} 0 & x \neq 0 \\ \dots & \end{cases}$$



$$\delta_y(x) \stackrel{\text{def}}{=\!=} \delta_0(x-y)$$

$$= \delta(x-y)$$

Correction:

Zhili Jiang (2025)