CPSC 536F Oct 10, 2025

What is the delta function?

Solve

$$w'(x) = x^2, \quad w(0) = 5$$

Rem: $w'(x) = f(x, w(x))$ like $w' = 5w$

$w(x) = e^{5x} + C$

Today's one simpler..

general solution

$$w'(x) = x^2$$

$$w(x) = \int x^2 dx + C$$

$$= x^3/3 + C$$

Then $w(0) = 5$ determines C

$$\left(x^3/3 + C\right)\Big|_{x=0} = 5$$

$$C = 5$$

$$w(x) = x^3/3 + 5$$

$w''(x) = x^3$, $\quad w(0) = 0$

$w(L) = 0$

$L \in \mathbb{R}$, fix

$$w'(x) = x^4/4 + C_1$$

$$w(x) = x^5/20 + C_1 x + C_0$$

$$\boxed{w(0) = 0} \implies 0^5/20 + C_1 \cdot 0 + C_0 = 0$$

$$C_0 = 0$$

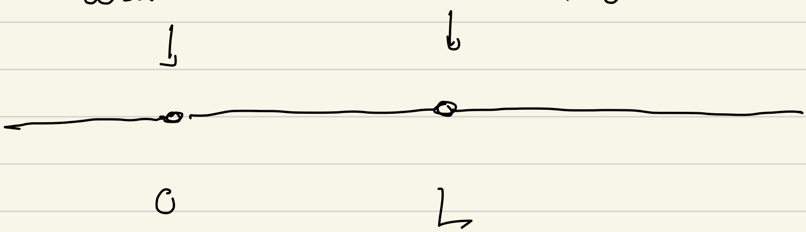
$$w(x) = x^5/20 + C_1 x$$

$$\boxed{w(L) = 0}$$

$$w(L) = L^5/20 + C_1 L = 0$$

$$C_1 L = -L^5/20, \quad C_1 = -L^4/20$$

$$w(x) = \frac{x^5}{20} + \left(-\frac{L^4}{20}\right)x$$



$$w''(x) = \sin(x), \qquad \begin{aligned} w(0) &= 0 \\ w(L) &= 0 \end{aligned}$$

$$w''(x) = e^{-(\sin(x))^5} + 25x^2 + (\tan(x))^{(\sin x)}$$

$$w'(x) = \int \qquad dx$$

Claim: To solve

$$w''(x) = f(x) \qquad \begin{aligned} w(0) &= 0 \\ w(L) &= 0 \end{aligned}$$

solution is

$$w(x) = \int_{t=0}^{t=L} f(t)\, G(t,x)\, dt$$

where

$G =$ "Green's function"

Problem:

$$G(t,x) = \text{simple stuff} + \underbrace{\mathrm{ReLU}(x-t)}_{\text{doesn't have 2 derivatives}}$$

$w''(x)$ when $w(x) = \int_{t=0}^{t=L} f(t)\, G(t,x)\, dx$

You'd like to say

$$w''(x) = \int_{t=0}^{t=L} f(t) \left(\frac{\partial}{\partial x}\right)^2 G(t,x)\, dt$$

$$= \int_{t=0}^{t=L} f(t)\, G_{xx}(t,x)\, dt$$

$w''(x) = f(x)$ $\quad$ $w(0) = 0$, $w(L) = 0$

$\tilde{w}(x) = \hat{\tilde{f}}(x)$ $\quad$ $\hat{w}(0) = 0$, $\hat{w}(L) = 0$

$$\underbrace{\left(6w(x)+5\tilde{w}(x)\right)''}_{W''(x)} = 5f(x)+6\tilde{f}(x)$$

The equation

$$w'' = f(x) \qquad \begin{array}{l} w(0)=0 \\ w(L)=0 \end{array}$$

is "linear in $f=f(x)$"

Also

$$f \longmapsto \int_0^L f(t)\,G(t,x)\,dx$$

is linear in f.

Generalized functions ③

comes

Generalized solution to an ODE ②

PDE

illustrate

ReLU(x) ← start ①

④ Green(x′, x)

$\quad\;\; ||$

Green(t, x) : our first

positive definite kernel funct

⑤ $e^{-|t-x|^2}$ second pos def ker function

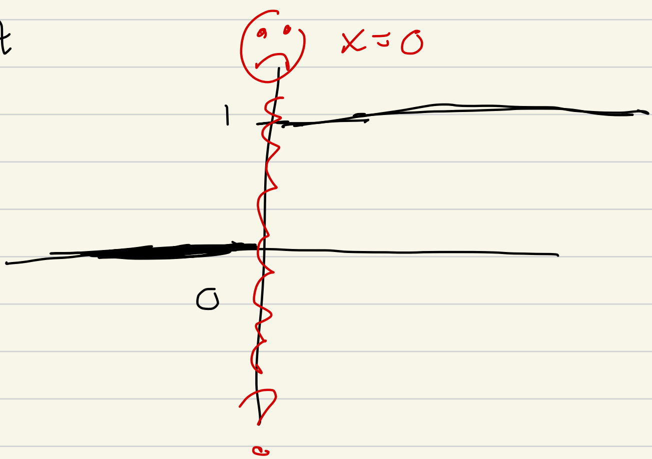
$e^{-|x'-x|^2}$

ReLU(x) : 

$$\stackrel{def}{=} \begin{cases} 0 & x \leq 0 \\ x & \geq 0 \end{cases}$$

ReLU' doesn't exist ... ☹

but



"Heaviside function"

ReLU(x)' at x = 0, OK,

But ReLU'(0) doesn't exist

First understand this ...

$$\mathrm{ReLU}(x) = w(x)$$

$$w'(x) = \mathrm{Heaviside}(x)$$

but not true at $x=0$

We say

$$w'(x) = f(x)$$

holds in a generalized sense ...

Take a "nice" $g = g(x)$, differentiable we insist that

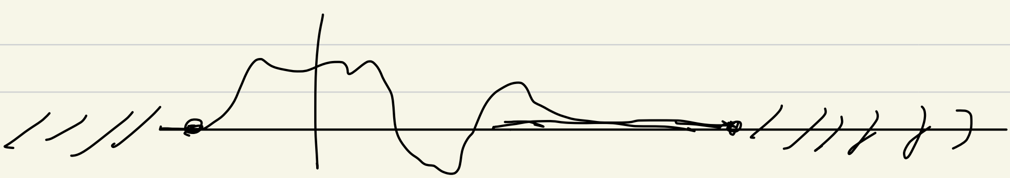
$$g(x)\, w'(x) = g(x)\, f(x)$$

this is of no help, but

$$\int_{-\infty}^{\infty} g(x)\, w'(x)\, dx = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx$$

for all g infinitely differentiable

g bounded support



i.e. for some $B \in \mathbb{R}$

$$g(x) = 0 \quad \text{if} \quad |x| \geq B$$

If ω, f are infinitely differentiable,

or just ω has one derivative

$$\int_{-\infty}^{\infty} g\,\omega' = -\int_{-\infty}^{\infty} g'\,\omega$$

$$\left[(g\omega)' = g'\omega + g\omega' \right]$$

$$\int_{-\infty}^{\infty} (g\omega)'(x)\,dx = (g\omega)(x)\Big|_{-\infty}^{\infty} = 0$$

If ω, f continuous, maybe

ω's derivative doesn't exist,

write

$$\omega' = f$$

if $\left[\begin{array}{l} \int g\omega' = \int gf \\ \quad \shortparallel \\ -\int g'\omega \end{array}\right]$

for all $g \in C_0^\infty(\mathbb{R})$ ← bounded supp, ∞-diff

$$\cancel{(*)} \int_{-\infty}^{\infty} (-g'w)(x)\,dx = \int_{-\infty}^{\infty} (gf)(x)\,dx$$

Now: claim

$$\text{ReLU}'(x) = \text{Heaviside}$$

in this <u>weak</u> <u>sense</u>

<u>WEAK</u> <u>SENSE</u>

$$\int_{-B}^{B} g'(x)\,\text{ReLU}(x)\,dx$$

$$= \int_{-B}^{B} g'(x) \begin{Bmatrix} 0 & x \leq 0 \\ x & x \geq 0 \end{Bmatrix} dx$$

$$= \int_0^B g'(x)\, x \, dx$$

(Rem: $C_0^\infty(\mathbb{R})$ sometimes called "bump function")

$$\int_{x=0}^{x=B} g'(x)\, x \, dx$$

$$= g(x)\, x \Big|_{x=-B}^{x=B} - \int_{x=0}^{x=B} \big(g(x)\cdot 1\big)\, dx$$

$$= 0 - \int_{x=0}^{x=B} g(x)\, dx$$

$$\int_{-B}^{B} -g'(x)\,\mathrm{ReLU}(x)$$

$$= \int_{x=0}^{x=B} g(x)\,dx \quad = \;???$$

$$\int_{x=-B}^{x=B} g(x)\,\mathrm{Heaviside}(x)\,dx$$

$$= \int_{x=-B}^{x=B} g(x) \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} dx$$

$$= \int_{x=0}^{x=B} g(x)\,dx \quad !!!$$

What about

$\mathrm{ReLu}''(x) = \mathrm{Heaviside}'(x) = \delta_0(x)$

not a classical function

Dirac delta function ~~1927~~ Gemini 1930 ChatGPT

Fourier, Poisson 1820's – 1830's

– .

To be continued