CPSC 536F Oct 8, 2025

Last time:

$k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ given by

$\Phi : \mathcal{X} \to \mathbb{R}^f$ and

$$k(x, x') = \Phi(x) \cdot \Phi(x'),$$

then

(1) k is pos semidef

(2) k is " definite $\Rightarrow |\mathcal{X}| \le f$.

$\mathcal{X}$ = infinite ... pos definite

" PDE (partial differential equation)

in $\mathbb{R}^n$, $n=1$ (i.e. ODE's)

(for ~~[illegible]~~ the reluctant)

Last time:

$$k(\cdot,\cdot)$$

$$= \begin{pmatrix} \begin{bmatrix}5\\1\end{bmatrix}\cdot\begin{bmatrix}5\\1\end{bmatrix} & \cdots & \begin{bmatrix}5\\1\end{bmatrix}\cdot\begin{bmatrix}\pi\\e\end{bmatrix} \\ \vdots & \begin{bmatrix}-17\\5\end{bmatrix}\cdot\begin{bmatrix}-17\\5\end{bmatrix} & \vdots \\ \vdots & \ddots & \begin{bmatrix}\pi\\e\end{bmatrix}\cdot\begin{bmatrix}\pi\\e\end{bmatrix} \end{pmatrix}$$

$$= \underbrace{\begin{bmatrix}5\\-17\\\pi\end{bmatrix}\begin{bmatrix}5 & -17 & \pi\end{bmatrix}}_{\text{top piece of dot product}} + \underbrace{\begin{bmatrix}1\\5\\e\end{bmatrix}\begin{bmatrix}1 & 5 & e\end{bmatrix}}_{\text{bottom piece}}$$

... homework ...

Claim: If $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ and $\hat{k} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ two kernel functions, then

(1) If $k, \hat{k}$ are symmetric, then so is $(k\hat{k})(x, x')$

$$\stackrel{\text{def}}{=} k(x, x')\,\hat{k}(x, x')$$

(2) If $k, \hat{k}$ are both pos semi-def then so is $k\hat{k}$

Why: ....

Lemma: If k is a kernel obtained

from $\Phi : \mathcal{X} \to \mathbb{R}^f$

(i.e. $k(x,x') = \Phi(x) \cdot \Phi(x')$)

and $\hat{k}$ from $\hat{\Phi} : \mathcal{X} \to \mathbb{R}^{\tilde{f}}$,

then

$$(k\hat{k})(x,x') = k(x,x')\tilde{k}(x,x')$$

<u>exercise</u>

kernel obtained

identify

from $\mathcal{X} \longrightarrow \mathbb{R}^f \otimes \mathbb{R}^{\hat{f}} \simeq \mathbb{R}^{f \cdot \tilde{f}}$

taking $x \longmapsto \Phi(x) \otimes \tilde{\Phi}(x)$

Pf: Homework ...

Quest: So what? Homework

We use: If K is pos semidef $n \times n$ matrix

$\Rightarrow \quad K = M^T M$ where

$$M \in \mathbb{R}^{f \times n} \quad (f \leq n)$$

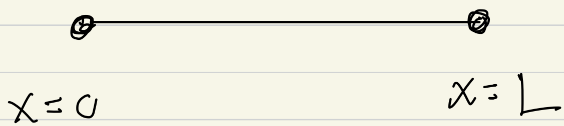
$$M^T \in \mathbb{R}^{n \times f}$$

PDEs in $\mathbb{R}^n$

$n = 1 \rightsquigarrow$ ODEs

Solve:

$\mathbb{R}^n$, $n=1$



$$w_{xx} = f \quad , \quad f \text{ given}$$

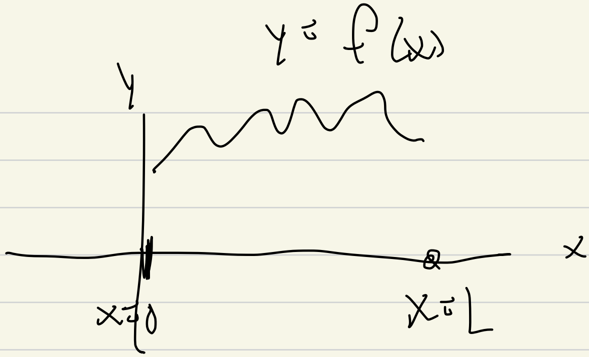
$$f : [0, L] \to \mathbb{R}$$

$$\left.\begin{aligned} w(0) &= 0 \\ w(L) &= 0 \end{aligned}\right\} \longleftrightarrow \text{Dirichlet conditions}$$

string held to 0 at $x=0$, $x=L$

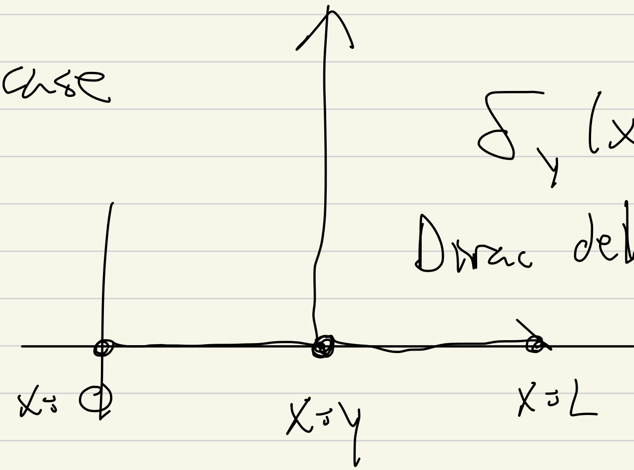
$$w_{xx} = \frac{d^2}{dx^2}(w)$$

$$w'' = f \quad \ldots$$



Special case

f :



$\delta_y(x) =$ Dirac delta funct

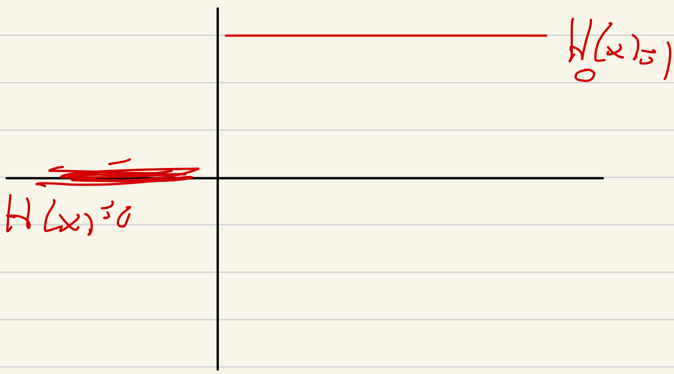
What is δ_0 = Delta function at $y = 0$

$\varepsilon > 0$

Say

$$\int_{x=-\infty}^{x=\infty} \delta_0(x)\, dx = \int_{x=-\varepsilon}^{x=+\varepsilon} \delta_0(x) = 1$$

$$\overbrace{\int_{t=-\infty}^{t=x} \delta_0(t)\,dt}^{H_0(x)} = \underbrace{\begin{cases} 0 & x<0 \\ 1 & x>0 \end{cases}}_{\text{Heaviside}}$$

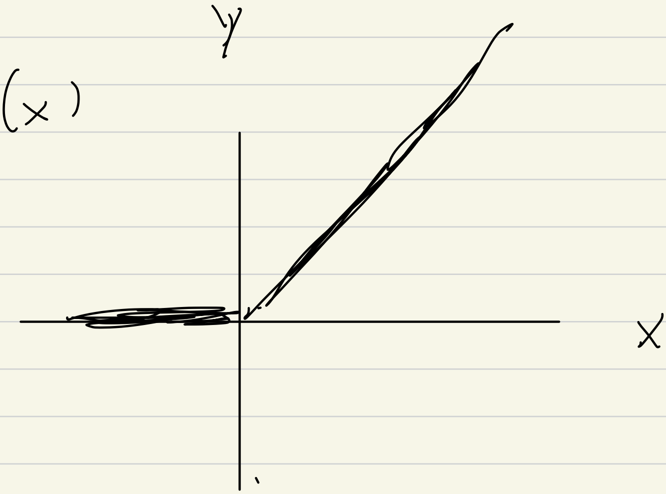


$H_0(x)$

(a step function)

$$\int_{t=-\infty}^{t=X} H(t)\,dt = \begin{cases} 0 & x<0 \\ X & x>0 \end{cases}$$

$y = \mathrm{ReLu}_0(x)$



Option trading.

$$w'' = \delta_0(x)$$

has $w(x) = \mathrm{ReLu}_0(x)$

as a solution

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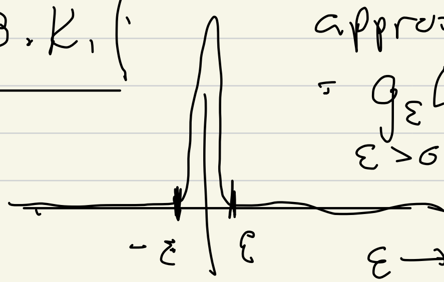
(1) Take this as O.K.

(2) Well: δ_0

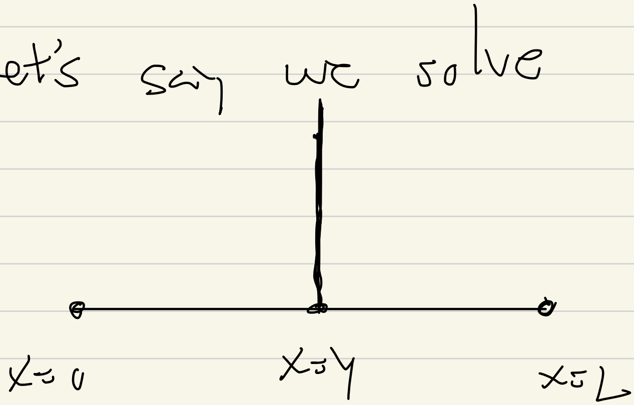


approx $= g_\varepsilon(x)$

$\varepsilon > 0$

$\varepsilon \to 0$

Let's say we solve



$w'' = \delta_y(x)$

solve:

$w'' =$

$$w'' = \delta_{L/10}\cdot 3 + \delta_{2L/10}\, 5 + \delta_{3L/10}\, 7$$

$$+ \cdots$$

Claim:

$$\boxed{w''(x) = f(x) \qquad w(0) = 0, \quad w(L) = 0}$$

Solution is

$$(*) \quad w(x) = \int f(t)\, \mathrm{Green}(t,x)\, dt$$

Green(t,x) solves

$$\left(\frac{d}{dx}\right)^2 \mathrm{Green}(t,x) = \delta_t(x)$$

$$\mathrm{Green}(t,0) = 0$$

$$\mathrm{Green}(t,L) = 0$$

In (*)

$$\boxed{\begin{aligned} w''(x) &= \int f(t)\,\delta_t(x)\,dx \\ &= f(x) \end{aligned}}$$

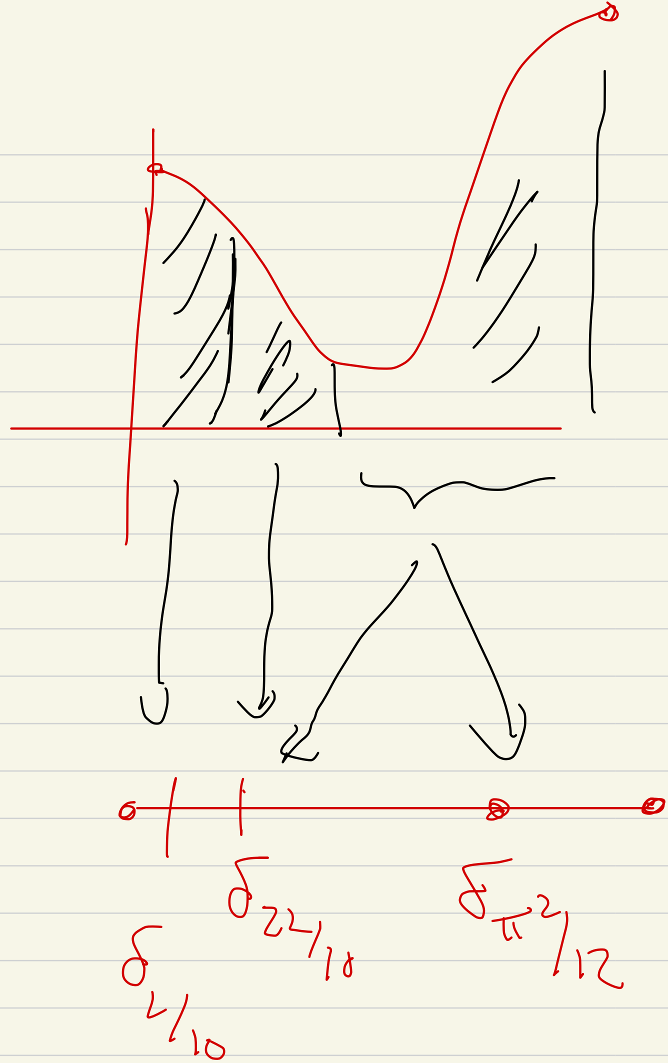
 !!! 

So

$$w''(x) = f(x)$$

If

$f(x)$:

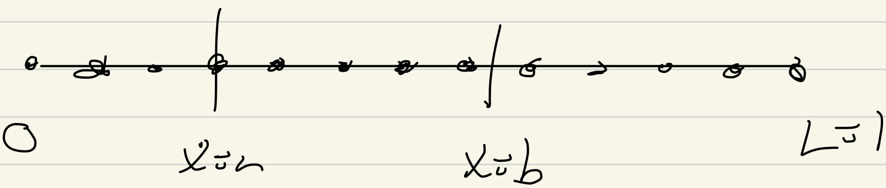


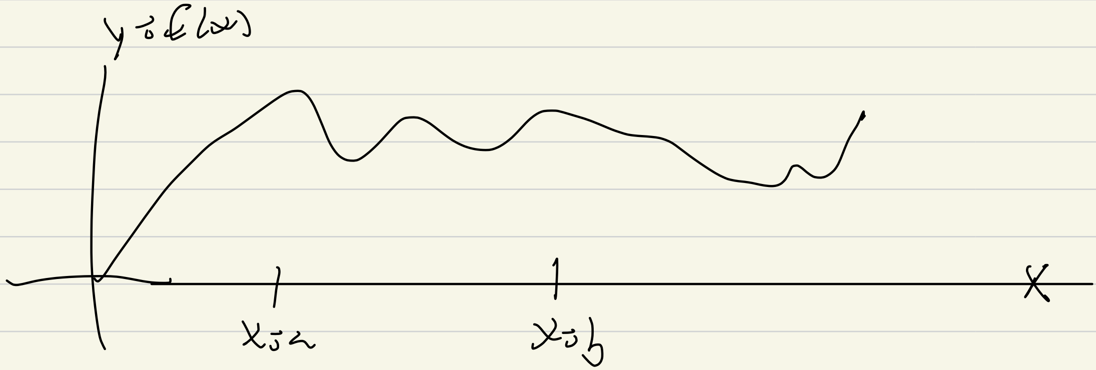
$\underset{\sim}{\mu}$

And ......

Green(x, y) is a pos definite kernel function !!! ...

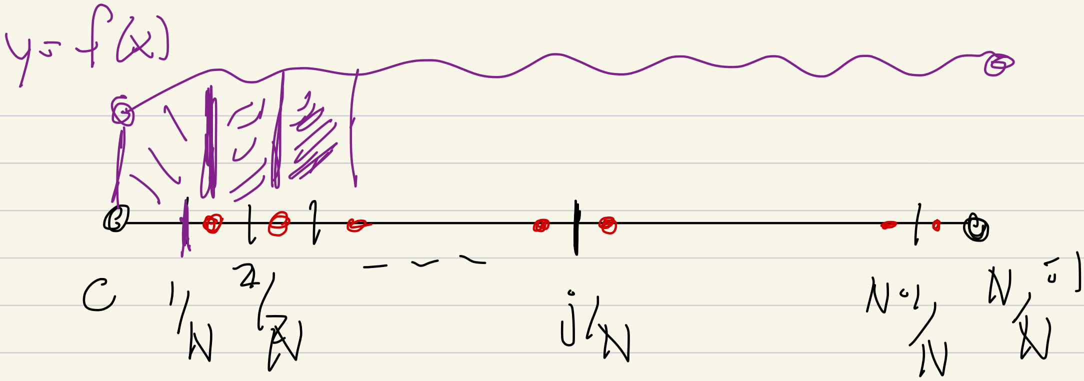
Class ends





$$\int_a^b f(x)\,dx = \text{something}$$

a, b vary



We have

$$\delta_{1/N}, \quad \delta_{2/N}, \quad \ldots, \quad \delta_{N-1/N}$$

$$c_1 = \int_0^{1.5/N} f(x)\,dx$$

$$c_2 = \int_{1.5/N}^{2.5/N} f(x)\,dx$$

$$\vdots$$

$$c_1 \delta_{1/N} + c_2 \delta_{2/N} + c_3 \delta_{3/N} + \cdots$$

$$\int_{\text{any } a}^{\text{any } b} f(t)\,dt$$

$$\int_{\text{any } a}^{\text{any } b} \left(c_1 \delta_{1/N} + c_2 \delta_{2/N} + \dots + c_{N-1} \delta_{N-1/N} \right)(t)\,dt$$

$$\approx \int_a^b f(t)\,dt$$

$$\left.\begin{array}{l} w'' = f \\ \tilde{w}'' = \tilde{f} \end{array}\right\}$$ Dirichlet,

if $\left| \int_a^b f(t)\,dt - \int_a^b \tilde{f}(t)\,dt \right| \le$

then