CPSC 536F Oct 6, 2025

Kernels: set theoretic map

$\Phi : \mathcal{X} \to \mathbb{R}^f$

f = # features (?)

Φ = "feature map"

f-dimensional

finite dimensional

Claim:

$$k(x, x') = \Phi(x) \cdot \Phi(x')$$

is "kernel trick" $\stackrel{?}{=}$ something easier to compute

— positive semidefinite

— positive definite iff

Φ is injective, and

$\Phi(\mathcal{X})$ is a set of linearly independent vectors in $\mathbb{R}^f$.

Definition: A positive semidefinite kernel function $k: \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is $\mathbb{R}^f$-representable ... blah blah blah

Thm: If k, k' are ... ... then

$$\tilde{k} = k(x,x')\, k'(x,x')$$

is ...

Survey results:

- Contiguous days (4)

  (2 added: 6 classes now on kernels)

- Different topics MWF (3)

Homework: blank / * 4

A/B/C 2

I'll make the blank/* file

Topics: 2, 2, 1 > 2 > 3,

3 > 1 > 2, 1, 2, 2

So far:

Definitions and product theorem:

Let $\mathcal{X}$ be a set, and

$\Phi : \mathcal{X} \to \mathbb{R}^f$ $\qquad f = 0, 1, 2, \ldots \in \mathbb{Z}_{\geq 0}$

Φ = "feature map"

f = "number of features"

Then

$$k(x, x') \stackrel{\text{def}}{=} \Phi(x) \cdot \Phi(x')$$

is called the <u>kernel (function)</u> <u>induced</u>

by Φ. Say that Φ, k are

f-dimensional, finite-dimensional.

Goal:

Prove: any such k is a

- symmetric, positive semidefine
- classify Φ s.t. k is definite

Prove: k, k' are arbitrary kernels that are pos semi def, then so is k, k'.

Rem: $\mathcal{X} = \mathbb{R}$, and

$$k(x,x') = e^{-|x-x'|^2 C}$$

$C > 0$, then k is pos definite.

Hence k is not induced from $\Phi : \mathbb{R} \to \mathbb{R}^f$

Thm: If $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is induced by $\Phi : \mathcal{X} \to \mathbb{R}^f$, then k is pos. semi-def.

Pf:

$$k(x,x') = \overbrace{\Phi(x) \cdot \Phi(x')}^{\text{makes sense as } \cdot \text{ in } \mathbb{R}^f}$$

$$= \Phi(x') \cdot \Phi(x) = k(x', x).$$

k is symmetric.

Let $x_1, \dots, x_m \in \mathcal{X}$.

[Think of $f = 5$, $m = 1000$.]

Then, for any $\alpha_1, \dots, \alpha_m \in \mathbb{R}$

$$\sum_{i,j=1}^{m} \alpha_i \alpha_j k(x_i, x_j) \overset{??}{\geq} 0$$

OR:

$$\begin{bmatrix} \vec{\Phi}(x_1)\cdot\vec{\Phi}(x_1) & \vec{\Phi}(x_1)\cdot\vec{\Phi}(x_2) & \cdots & \vec{\Phi}(x_1)\cdot\vec{\Phi}(x_m) \\ \vec{\Phi}(x_2)\cdot\vec{\Phi}(x_1) & & & \vdots \\ \vdots & & & \vdots \\ & \cdots & & \vec{\Phi}(x_m)\cdot\vec{\Phi}(x_m) \end{bmatrix}$$

[If $f = 5$, $m = 1000$, $\uparrow$ is a 1000×1000 matrix]

But....

$$\sum_{i,j} \alpha_i \alpha_j k(x_i, x_j) = \ldots ?$$

(since $k(x,x') = \Phi(x) \cdot \Phi(x')$ is bilinear)

$$\overbrace{\left(\alpha_1 \Phi(x_1) + \ldots + \alpha_m \Phi(x_m)\right)}^{\vec{v}}$$

$$\cdot \underbrace{\left(\alpha_1 \Phi(x_1) + \ldots + \alpha_m \Phi(x_m)\right)}_{\vec{v}}$$

$$= \sum \alpha_i \alpha_j \Phi(x_i) \Phi(x_j)$$

is ≥ 0

So k is pos semidefinite.

Rem: $\vec{v}$ is rather strange thing....

Example:

Let $\mathcal{X} = \mathbb{R}$, let

$\Phi : \mathcal{X} \to \mathbb{R}^3$

$$x \longmapsto \begin{bmatrix} 1 \\ 2 \\ -17 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -2 \\ 17 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \cancel{2} \\ 17 \end{bmatrix} = 294$$

For any $x_1, \ldots, x_m$,

$$\begin{bmatrix} \underbrace{k(x_1, x_1)}_{= \Phi(x_1) \cdot \Phi(x_1)} & \cdots & & \vdots \\ \vdots & k(x_i, x_j) & & \vdots \\ \vdots & & & \\ \cdots & \cdots & & k(x_m, x_m) \end{bmatrix} = \begin{bmatrix} 294 & 294 & \cdots \\ \vdots & & \\ 294 & \cdots & \end{bmatrix}$$

$\Phi(x_i) \cdot \Phi(x_j)$

$$= 294 \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & & & \\ 1 & \cdots & & 1 \end{bmatrix}$$

λ : 294 mult 1, 0 mult $m-1$

Example 2:

Let $\Phi : \mathbb{R} \to \mathbb{R}$ be

$\Phi(x) = x$.

$\Phi(0) \cdot \Phi(0) = 0 \cdot 0 = 0$

$k(0,0) = 0$

$x_1 = 0$, $\left[k(x_1, x_1) \right] = [0]$

So $\alpha_1 = 1$, $x_1 = 0$

$\alpha_1 \alpha_1 k(x_1, x_1) = 0$

So Φ is not pos definite kernel.

So... say you have a positive definite

$$k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$$

then $k(x, x) > 0$ for all $x \in \mathcal{X}$

pos semidef,

$k(x, x) \geq 0$ " " "

Continuing $\Phi(x) = x$:

$x_1 = 1$, $x_2 = 2$, $x_3 = 3$, then

$$\begin{bmatrix} k(x_1, x_1) & \cdots & \\ \vdots & & \\ & & k(x_3, x_3) \end{bmatrix}$$

$k(x, x')$

$= \Phi(x) \cdot \Phi(x')$

is
$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

Nullspace $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ has dim 2.

Example 3: $\mathcal{X} : \mathbb{R} \to \mathbb{R}^2$

where
$$\mathcal{X}(0) = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$
$$\mathcal{X}(1,3) = \begin{bmatrix} -17 \\ 5 \end{bmatrix}$$
$$\mathcal{X}(2) = \begin{bmatrix} \pi \\ e \end{bmatrix}$$
(transcendental irrational)

$x_1 = 0, \quad x_2 = 1.3, \quad x_3 = 2 ;$

$k(\cdot, \cdot)$

$$= \begin{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \end{bmatrix} & \cdots & \begin{bmatrix} 5 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \pi \\ e \end{bmatrix} \\ \vdots & \begin{bmatrix} -17 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -17 \\ 5 \end{bmatrix} & - \\ - & \cdot & \begin{bmatrix} \pi \\ e \end{bmatrix} \cdot \begin{bmatrix} \pi \\ e \end{bmatrix} \end{bmatrix}$$

Choose $\alpha_1, \alpha_2, \alpha_3$ s.t.

$$\alpha_1 \underbrace{\begin{bmatrix} 5 \\ 1 \end{bmatrix}}_{\vec{u}_1} + \alpha_2 \underbrace{\begin{bmatrix} -17 \\ 5 \end{bmatrix}}_{\vec{u}_2} + \alpha_3 \underbrace{\begin{bmatrix} \pi \\ e \end{bmatrix}}_{\vec{u}_3} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left(\alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 + \alpha_3 \vec{u}_3 \right)$$

$$\cdot \left(\text{same thing} \right)$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$$