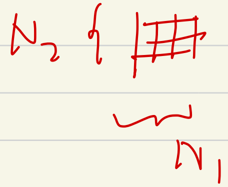
CPSC 536F Oct 3, 2025

$n = N_1 N_2$



- Last time:

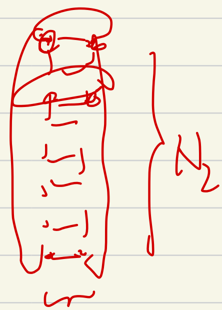
  $N_1 \times N_2$ cyclic grid

$$\lambda_1 = 4, \quad \lambda_2 \approx 4 - \frac{C_1}{(\max(N_1, N_2))^2}$$

e.g. $N_1 = N_2 = \sqrt{n}$

Spectral gap

$$\lambda_1 - \lambda_2 \approx \frac{C_1}{n}$$



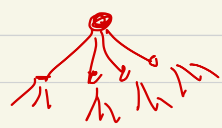
e.g. $N_1 = 3$, $N_2 = \frac{n}{3}$

$$\lambda_1 - \lambda_2 \approx \frac{C_1}{(n/2)^2} \approx \frac{4C_1}{n^2}$$

Geometrically ... why

$$\frac{c_1}{n} \quad \text{vs.} \quad \frac{4c_1}{n^2}$$

Random 4-graph:



tree

Claim:

(Fr) $\lambda_1 - \lambda_2 \approx 4 - 2\sqrt{3}$

Geometry = "tree-like" ("Birthday Paradox")

G d-regular: ∂ = incidence matrix

$$\Delta_G = dI - A_G = \partial \partial^T$$

$\lambda_1(A_G) = d$

$$0 = \lambda_1(\Delta_G) \le \dots \le \lambda_n(\Delta_G) = 2d$$

$\Delta_G :=$ "graph Laplacian"

Today: Back to kernels, ridge regression, ...

$K \in \mathbb{R}^{n \times n}$ is

- symmetric: $K^T = K$ $\Rightarrow$ real eigenvalues, eigenvectors

- pos semidef:

$\forall \vec{v} \in \mathbb{R}^n, \quad \vec{v}^T K \vec{v} \geq 0$

- pos def:

$\vec{v}^T K \vec{v} = 0 \iff \vec{v} = \vec{0}$

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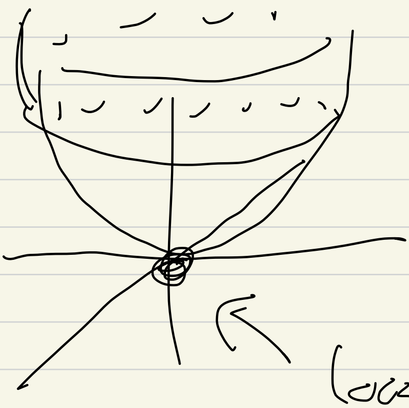
e.g. $f(\vec{x}) = x_1^2 + x_2^2 + x_3^2$, $K = (\mathrm{Hess}(f))(0)$

(local min)

$$f(x_1, x_2) = x_1^2 + x_2^2$$

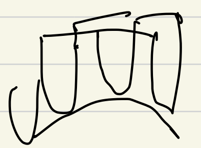
$$\text{Hess}(f) = \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} \\ f_{x_1x_2} & f_{x_2x_2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \leftarrow$$ sym
eig's
$= 2, 2$



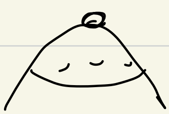
local minimum

saddle



$x_1^2 - x_2^2$

λ's
$2, -2$

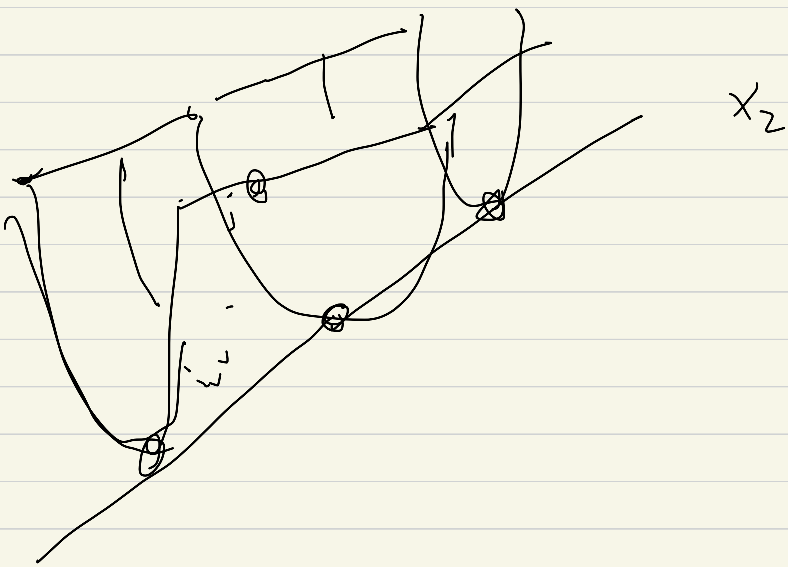


λ's
$-2, -2$

"Degenerate"

$$f(x_1, x_2) = x_1^2 + 0 \cdot x_2^2$$

$$+ \Big(\underbrace{x_1^3,\ x_1^2 x_2,\ \ldots}_{\text{higher order}} \Big)$$



Kernel: $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$

- symmetric
- pos semi-def
- pos def

restrict to finite $\mathcal{X}' \subset \mathcal{X}$

⋮

Example:

$\Phi : \mathcal{X} \longrightarrow \mathbb{R}^m$ "features"

$$k(x, x') \stackrel{\text{def}}{=} \Phi(x) \cdot \Phi(x')$$

$$\boxed{\text{\underline{Thm}: If } |\mathcal{X}| > m \text{, then } k \text{ is \underline{not} positive definite.}}$$

$$\mathcal{X} = \mathbb{R}^2 \qquad \vec{x} = (x_1, x_2), \ \vec{x}' = (x_1', x_2')$$

$$k(\vec{x}, \vec{x}') \overset{def}{=}$$

$$(1, \sqrt{2}\, x_1, \sqrt{2}\, x_2, x_1^2, \sqrt{2}\, x_1 x_2, x_2^2)$$

$$\cdot (1, \sqrt{2}\, x_1', \sqrt{2}\, x_2', (x_1')^2, \sqrt{2}\, x_1' x_2', (x_2')^2)$$

$$k(\vec{x}, \vec{x}) =$$

kernel trick

$$(1 + \vec{x} \cdot \vec{x})^2$$

$$k(\vec{x}, \vec{x}') = (1 + \underbrace{\vec{x} \cdot \vec{x}'}_{\mathbb{R}^2})^2$$

Kernel function

$\mathcal{X}$ is a finite set

$\mathcal{X} = \{1, \dots, n\}$

kernel function on $[n]$

$\longleftrightarrow$ $n \times n$ matrix

Since

$$\begin{bmatrix} k(1,1) & k(1,2) & \cdots & k(1,n) \\ k(2,1) & k(2,2) & \cdots & \vdots \\ \vdots & & & \vdots \\ k(n,1) & \cdots & \cdots & k(n,n) \end{bmatrix} = K$$

$k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$, $|\mathcal{X}| < \infty$

is symmetric if

$$\mathbb{K} = \begin{bmatrix} k(x_1, x_1) & \cdots & k(x_1, x_n) \\ \vdots & & \\ k(x_n, x_1) & \cdots & k(x_n, x_n) \end{bmatrix}$$

(where $\mathcal{X} = \{x_1, \ldots, x_n\}$, $|\mathcal{X}| = n$)

$\mathbb{K}$ is symmetric

k is pos semidef if $\mathbb{K}$ ...

k is pos def if $\mathbb{K}$ ...

$k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ if $|\mathcal{X}| = \infty$,

is

$$\left.\begin{array}{ll} \text{symm.} & \iff \\ \text{pos semidef} & \iff \\ \text{pos def} & \iff \end{array}\right\} \text{for all finite } \mathcal{X}' \subset \mathcal{X}, \quad \left.\begin{array}{l} \text{symm} \\ \text{pos semidef} \\ \text{pos def} \end{array}\right\} \begin{array}{l} \text{for the} \\ \text{restriction} \\ \mathcal{X}' \times \mathcal{X}' \to \mathbb{R} \end{array}$$

Other:

k symmetric if $k(x, x') = k(x', x)$
for all $x, x' \in \mathcal{X}$

k is pos semidef if $\forall\, x_1, \ldots, x_m \in \mathcal{X}$

for all $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$

$$\sum_{i,j=1}^{m} \alpha_i \alpha_j k(x_i, x_j) \geq 0$$

pos def iff $\uparrow$ equality

iff $\alpha_1 = \alpha_2 = \dots = \alpha_m = 0$

Example:

$\Phi : \mathcal{X} \longrightarrow \mathbb{R}^f$ (f = features)

Then

$$k(x, x') = \Phi(x) \cdot \Phi(x')$$

then: k is pos semidef

If k is pos def, then $|\mathcal{X}| \leq f$

(1) Applications

Which do you want more of?

- Graphs: Expanders
- Graphs: Symbolic Dynamics, Shannon Entropy, RLL codes
- Markov chains: Mixing Time

(2) Applications

- SVD, PCA, Low Rank Approximation
- Regression, Ridge Regression

(3) Kernel Viewpoint & Kernel Methods

kernel regression

HW Questions:
- A, B, C
- Reg, *
- No designation

Do you want:

Monday: Graph Thy

W, F: Kernel

or — —

OR

6 classes kernel methods,
thy

Other feedback?

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Subject: — — 536F . . . . . .