CPSC 536F Oct 1, 2025

Coming soon: if $K \in \mathbb{R}^{n \times n}$ is positive semi-definite, then it's easy to see the same holds for $p(K)$ for any p taking $\mathbb{R}_{\geq 0}$ to itself... NOT EASY: so is the entry-wise square of K

Meaning of "spectral gap" in adjacency eigenvalues in d-regular graphs.

G is a (simple) graph

Adjacency eigenvalues: A_G symmetric:

$$\lambda_n(A_G) \le \ldots \le \lambda_1(A_G)$$

<u>Thm</u>: Say G is d-regular. Then

① $\lambda_1 = d$

② Multiplicity of d is # conn comps of G

(hence G is connected $\iff \lambda_2 < \underset{\overset{\shortparallel}{d}}{\lambda_1}$)

Similar interpretation of multiplicity of $-d$ and # of bipartite components

Admin:

CPSC 531F from Spring 2021

has many interesting examples

Our homepage has a link

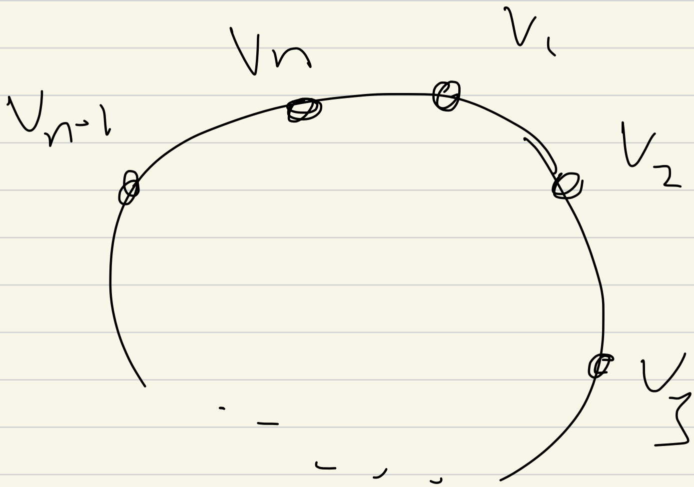
to

- The 2021 CPSC 531F homepage
- The notes from then
- The problems assigned
  - Problems/homework: A, B, C

Last term: CPSC 531F Spring 2025

Problems – no gradations

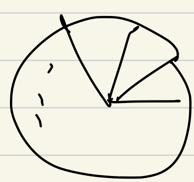
The n-cycle:



$n \geq 3$ (since we insist that G is simple)

$$\vec{u} = \begin{bmatrix} 1 \\ \zeta \\ \zeta^2 \\ \vdots \\ \zeta^{n-1} \end{bmatrix}$$

where $\zeta^n = 1$

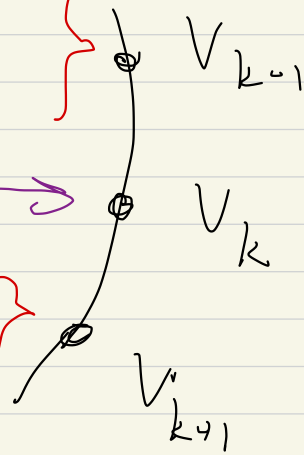


$\zeta = e^{(2\pi i/n) m}$

$\vec{u}(v_{k-1}) = \zeta^{k-1}$

$\vec{u}(v_k) = \zeta^k$

$\vec{u}(v_{k+1}) = \zeta^{k+1}$



$$(A_G \vec{u})(v_k) \overset{\text{by what } A_G \text{ is}}{=} \vec{u}(v_{k-1}) + \vec{u}(v_{k+1})$$

Here $= \zeta^{k-1} + \zeta^{k+1}$

$$= \zeta^k(\zeta^{-1} + \zeta) = \zeta^k(\bar{\zeta} + \zeta)$$

$$\vec{u}(v_k) = \zeta^k$$

$$(A_G \vec{u})(v_k)$$

$$= (\bar{\zeta} + \zeta)\, \vec{u}(v_k)$$

$$\zeta_m = e^{2\pi i m/n}, \qquad m = 0, 1, \ldots, n-1$$

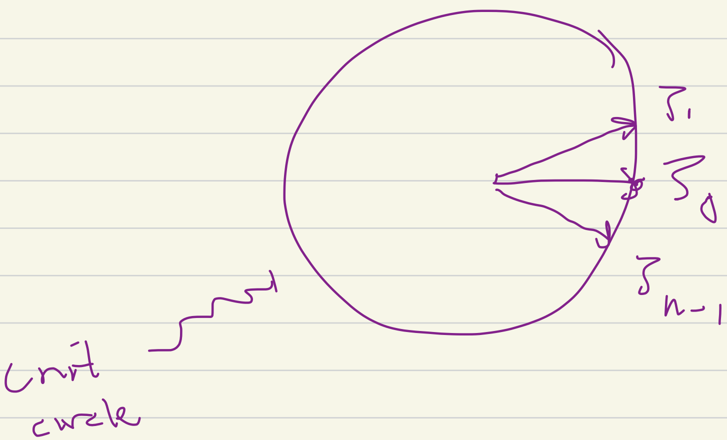
$$\bar{\zeta} = \cos(2\pi m/n) - i \sin(2\pi m/n)$$

$$\zeta = \text{''} \quad + \quad \text{''}$$

$$A_G \vec{u}_m = \vec{u}_m \left(2 \cos 2\pi m / n \right)$$

$$\vec{u}_m = \begin{bmatrix} 1 \\ \zeta_m \\ \zeta_m^2 \\ \vdots \end{bmatrix}, \quad \lambda_m = 2\cos\left(\frac{2\pi m}{n}\right)$$

$$m = 0, \quad \lambda_0 = 2$$



$m=1$:

$$\lambda_1 = 2\cos\left(\frac{2\pi}{n}\right)$$

$$= 2\left(1 - \left(\frac{2\pi}{n}\right)^2 \frac{1}{2} + (\ \ )^4 \frac{1}{4!} + \dots\right)$$

$$= 2 - (2\pi)^2 \frac{1}{n^2} + O\left(\frac{1}{n^4}\right)$$

here $O\left(\frac{1}{n^4}\right)$ means some

function $f = f(n)$ s.t. $\exists\, C$ s.t.

for large n, $|f(n)| \leq \frac{C}{n^4}$.

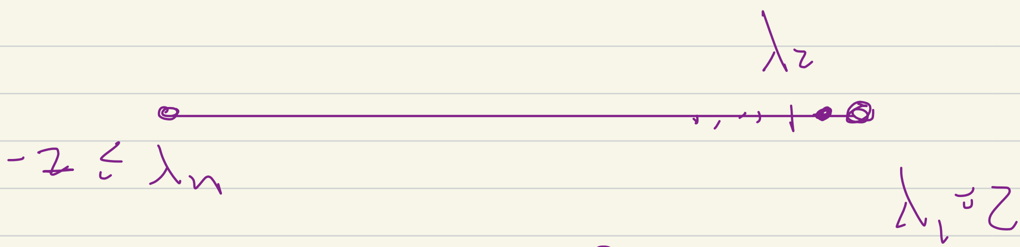
So, for a cycle of length n:

$$-2 \le \lambda_n(A_G) \ldots \lambda_2 < \lambda_1(A_G) = 2$$

But

$$\lambda_2 = 2 - \frac{c_1}{n^2} + O\left(\frac{1}{n^4}\right)$$

where $c_1 = (2\pi)^2$



$$\lambda_1 - \lambda_2 \approx \frac{c_1}{n^2}$$

Note $\lambda_n = -2$

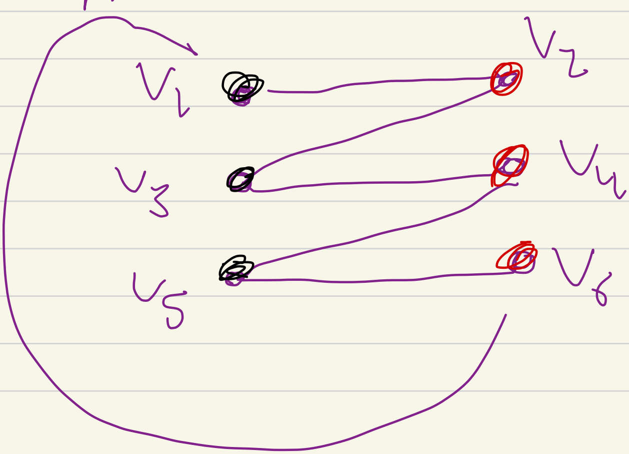
iff $\cos\left(\frac{2\pi m}{n}\right) = -1$

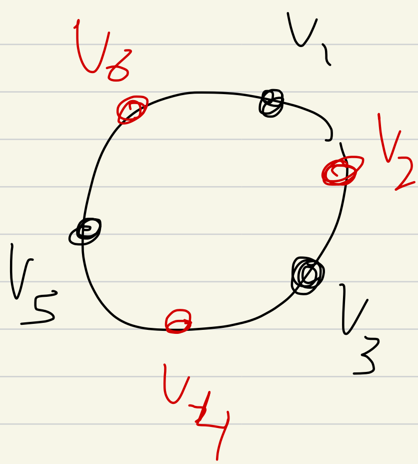
for some m

If n even, $m = \frac{n}{2}$

$\cos(2\pi m) = \cos(\pi) = -1$

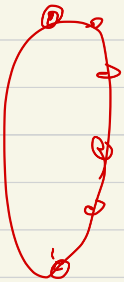
$n = 6$



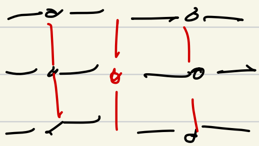


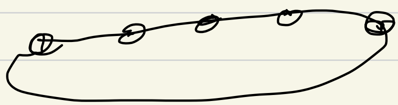
What about H

vertices is $N_1 N_2$



Cycle length N_2





G

Cycle length N_1

λ's of $A_{G \times H}$

= sum of λ's of A_G

λ's of A_H

λ's A_G are $2, \; 2 - \frac{c_1}{N_1^2} + O\left(\frac{1}{N_1^4}\right), \; \ldots$

close to -2

$= -2$ iff N_1 is even

λ's A_H $\quad 2, \; 2 - \frac{c_1}{N_2^2} + \text{small}, \ldots$

(-2 iff N_2 is even)

get $4, \quad 4 - \frac{c_1}{N_2^2} + \text{small}$

$4 - \frac{c_1}{N_1^2} + \text{small} \; \ldots$

(-4 iff N_1, N_2 both even)

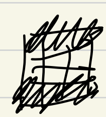
$n = N_1 N_2 = \#$ vertices in $G \times H$

So

$$\lambda_1 = 4, \quad \lambda_2 > 4 - \frac{c_1}{(\sqrt{n})^2} + \text{small}$$

$$= 4 - \frac{c_1}{n} + \text{small}$$

$$\lambda_1 - \lambda_2 \gtrsim \frac{c_1}{n} + \text{small}$$



Cycle

$$\lambda_1 - \lambda_2 \approx \frac{c_1}{n^2}$$