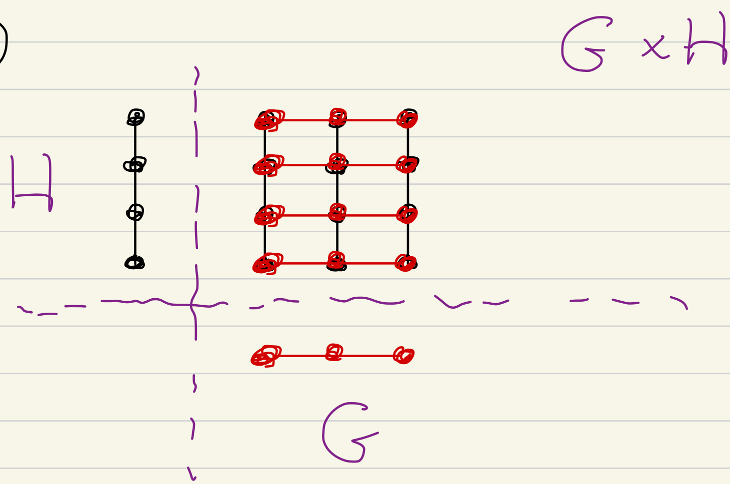
CPSC 536F Sept 29, 2025

Good expansion vs. bad expansion:

- Expansion is an umbrella term for graphs that are "well interconnected."

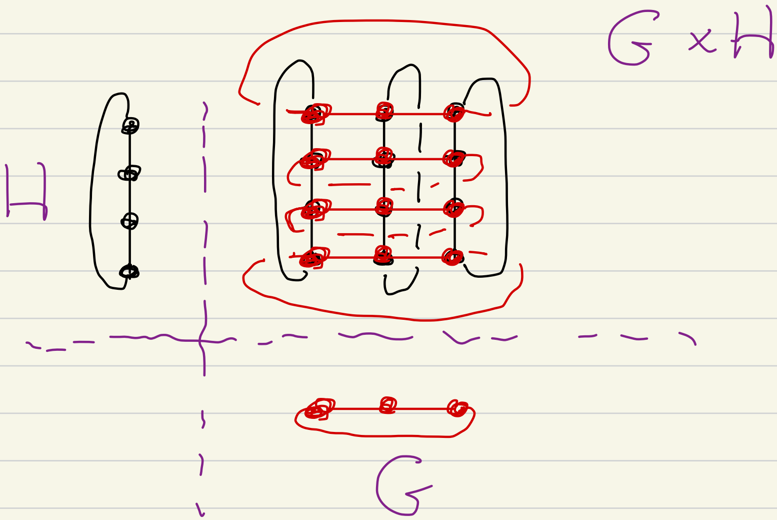
- Compare:

(1)



"Grid graph"

② 4-regular cyclic grid graph



"Cyclic Grid graph"

③ Random 4-regular graph

"Spectral gap"

④ Some other "expanders"

- Then return to kernels

(theory, building them, ...)

Think about the following:

we say

$$k : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}$$

for a set $\mathcal{X}$ is a

symmetric kernel: $k(x, x') = k(x', x)$

positive semidefinite if given any

finite $\mathcal{X}' \subset \mathcal{X}$,

$k|_{\mathcal{X}' \times \mathcal{X}'}$ as a matrix is positive semidefinite.

<u>Claim:</u> If $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is a positive semidefinite kernel (symmetric), then

$$k^2(x, x') \overset{\text{def}}{=} \big(k(x, x')\big)^2$$

is also positive semidefinite.

Proof: use ⊗ in a clever way... Very short proof...

Far from trivial if you don't know ⊗

Claim: If

K is symm, positive semidef

then so is

$\tilde{K}$ whose entries are

$$\tilde{K}(x, x') \stackrel{def}{=} \left(K(x, x')\right)^2$$

$$\begin{bmatrix} 1.1 & \pm 1 \\ \pm 1 & 1.1 \end{bmatrix} \Rightarrow \begin{bmatrix} (1.1)^2 & 1^2 \\ 1^2 & (1.1)^2 \end{bmatrix}$$

$K\vec{v} = \lambda\vec{v}$, $\quad$ K entry-wise square $\vec{v}$??

Rem:

$$K\vec{v} = \lambda\vec{v}$$

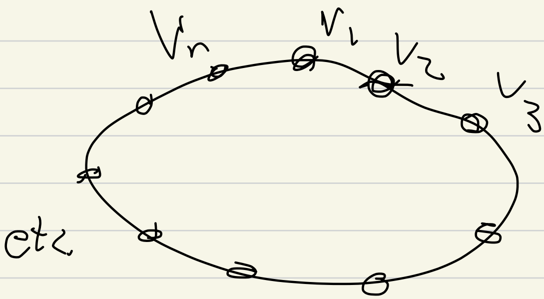
$$K^2\vec{v} = KK\vec{v}$$

$$= K(\lambda\vec{v})$$

$$= \lambda K\vec{v}$$

$$= \lambda^2\vec{v}$$

Expansion:

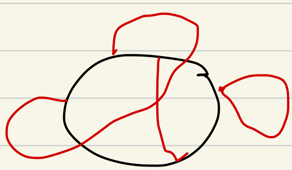


Cycle of length n (the graph)

$$V = \{v_1, v_2, \ldots, v_n\}$$

$$E = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$$

Warning: You can speak of a cycle of length n in a graph



(a subgraph of a graph)

A_G, $G =$ cycle of length n

$$\begin{array}{cc} & \begin{matrix} 1 & 2 & 3 & & & \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ \\ \\ \\ \end{matrix} & \begin{bmatrix} 0 & 1 & & & & 1 \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & & & 1 \\ 1 & & & & 1 & 0 \end{bmatrix} \end{array}$$

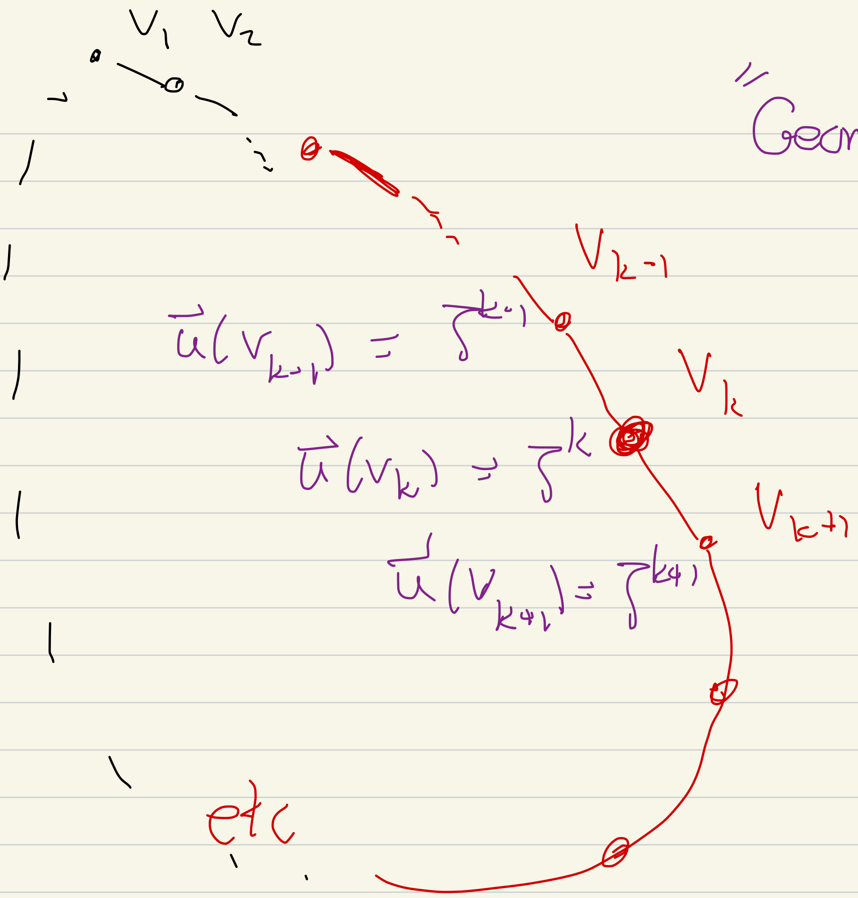
circulant

$\vec{u}$

$$A_G \begin{bmatrix} 1 \\ \zeta \\ \zeta^2 \\ \zeta^3 \\ \vdots \\ \zeta^{n-1} \end{bmatrix},$$

where $\zeta^n = 1$

n^{th} root of unity



$\vec{u}(v_{k-1}) = \zeta^{k-1}$

$\vec{u}(v_k) = \zeta^k$

$\vec{u}(v_{k+1}) = \zeta^{k+1}$

"Geometric view" of local picture, part of a cycle of length n

$$\left(A_G \begin{bmatrix} \vdots \\ \zeta_{k-1} \\ \zeta_k \\ \zeta_{k+1} \\ \vdots \end{bmatrix} \right)_{k\text{th component}} = \zeta^{k-1} + \zeta^{k+1}$$

Remark: If G is a graph,

A_G is a $V_G \times V_G$ matrix

an eigenvector $\vec{u} : V_G \to \mathbb{R}$

it's more canonical to think

(even if $V_G = \{v_1, \ldots, v_n\}$)

$$\vec{u} : \underset{v \in V_G}{\begin{bmatrix} \vec{u}(v) \\ \vec{u}(v) \\ \vec{u}(v) \end{bmatrix}}$$

natural:

$$\begin{bmatrix} \vec{u}(v_1) \\ \vec{u}(v_2) \\ \vdots \\ \vec{u}(v_n) \end{bmatrix}$$

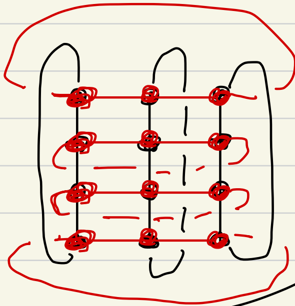
If $$A_G : V_G \times V_G \to \mathbb{R}$$

is $n \times n$ matrix

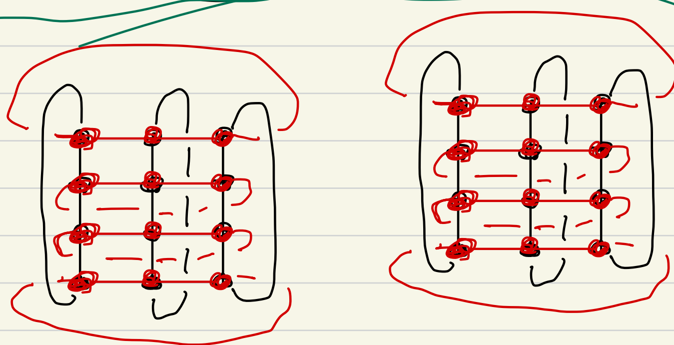
$$\begin{bmatrix} A_G(v_1, v_1) & A_G(v_1, v_2) & \cdots \\ A_G(v_2, v_1) & \cdots & \\ \vdots & & \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$ where $u_k = \vec{u}(v_k)$

mult of $\lambda = 4$:

$G_1 =$ 

$\leftarrow$ mult 1

G_2 

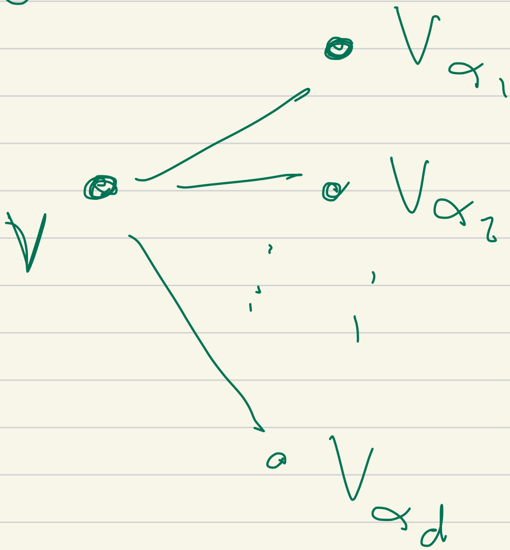
mult 2

4-regular

$\lambda_2 < \lambda_1 = 4$

$\lambda_2 = 4 = \lambda_1 = 4$

$\deg(v) = d$



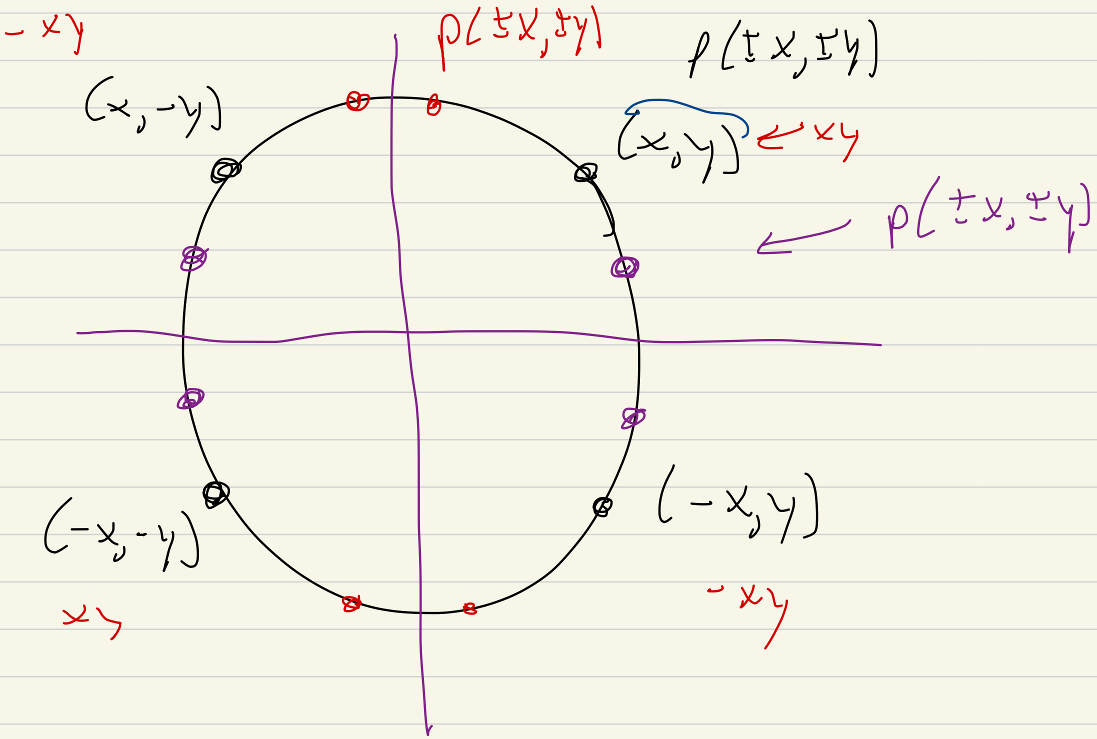
$$\left(A_G \vec{u}\right)(v)$$

$$= \vec{u}(v_{\alpha_1}) + \vec{u}(v_{\alpha_2}) + \dots + \vec{u}(v_{\alpha_d})$$

$$= \sum_{w \sim u} \vec{u}(w)$$

$w \sim u$ means w is adjacent to u

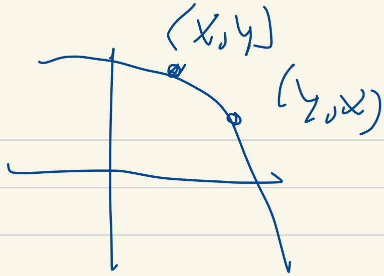
Class ends . . .



$p(x,y) = p(x,-y) = p(-x,-y) = p(-x,y)$

Claim: $E_p[x] = 0$, $E_p[y] = 0$, $E_p(xy) = \gamma$

$$E_p[x^2] = E_p[y^2]$$



(Zhili Jiang)

$$\{(x,y) : F(x,y) - 1 = 0\}$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \quad \text{ellipse}$$

$$(F(x,y) - 1)\,G(x,y) = 0$$

on this ellipse

$G(x,y)$ is never 0

$$\left(\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 1\right)\left(e^{\cancel{x}} + e^{\cancel{y}} + (\cancel{xy})^{20}\right) = 0$$

$$\underbrace{\left(\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 1\right)\left(e^x + e^y + (xy)^{20}\right) + 1}_{\tilde{F}(x,y)} = 1$$