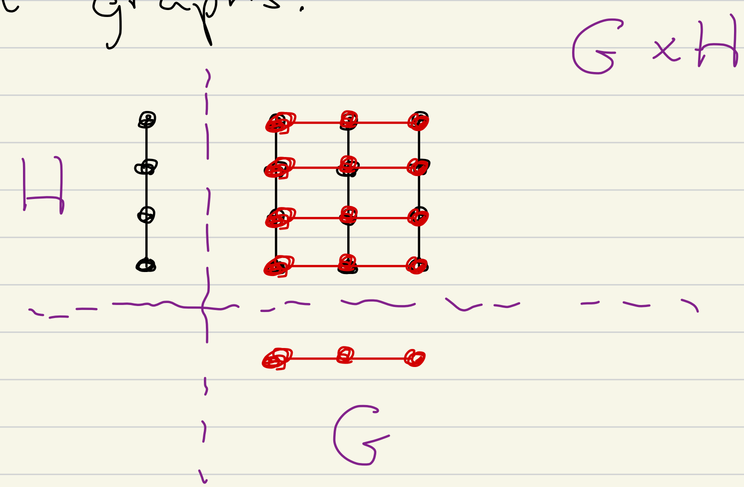
CPSC 536F Sept. 26, 2025

Last time:

- Definition: Cartesian product of graphs.



Def: $G \times H$ is blah blah blah

$$A_{G \times H} = A_G \otimes I_{V_H} + I_{V_G} \otimes A_H$$

- Fibre product of graphs

$G \otimes H$, $A_{G\otimes H} = A_G \otimes A_H$

- "Good expansion" versus "Bad expansion"

Admin: We now have a

\# problem.

Option worth doing

* → straightforward

H  $A_H = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

G  $A_G = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

//

$\mathbb{B}^1$

$$A_{G\times H} \quad \begin{array}{c} \\ (v_1,u_1) \\ (v_1,u_2) \\ (v_1,u_3) \\ (v_2,u_1) \\ (v_2,u_2) \\ (v_2,u_3) \end{array} \begin{array}{c} \begin{array}{cccccc} v_1u_1 & v_1u_2 & v_1u_3 & v_2u_1 & v_2u_2 & v_2u_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & & & & & \\ & 1 & & & ? & ? \\ & & 1 & & & \end{bmatrix} \\ \begin{array}{cccccc} v_1u_1 & v_1u_2 & v_1u_3 & v_2u_1 & v_2u_2 & v_2u_3 \end{array} \end{array}$$

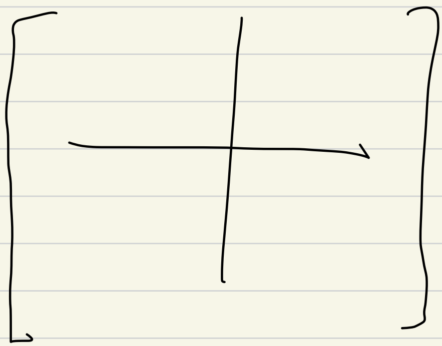
Maybe... $A_{G\times H} = A_G \otimes I_{V_H} + I_{V_G} \otimes A_H$

$$I_{V_G} \otimes A_H$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \left[\begin{array}{c|c} A_H & 0 \\ \hline 0 & A_H \end{array}\right]$$

=



$M \otimes N \rightarrow$ looks ex M $\begin{bmatrix} 3 & 7 & 1 \\ 2 & 1 & 9 \end{bmatrix}$

put the N blocks into it

M is $\begin{bmatrix} 3 & 7 & 1 \\ 2 & 1 & -9 \end{bmatrix}$

$M \otimes N$

$$= \begin{bmatrix} 3N, & 7N, & N \\ 2N, & 1N, & -9N \end{bmatrix}$$

matrix on right "goes into"
" " left

(Could also do
left "goes into"
right — —

$$I_{V_G} \otimes A_H$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \left[\begin{array}{c|c} A_H & 0 \\ \hline 0 & A_H \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & & & \\ 1 & 0 & 1 & & 0 & \\ 0 & 1 & 0 & & & \\ \hline & & & 0 & 1 & 0 \\ & 0 & & 1 & 0 & 1 \\ & & & 0 & 1 & 0 \end{array}\right]$$

$$A_G \otimes I_{V_H}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ \hline \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} & & 0 & \end{array}\right]$$

Sum:

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\left.\begin{array}{l} A_G \vec{w} = \lambda \vec{w} \\ \\ A_H \vec{x} = \mu \vec{x} \end{array}\right\} \quad \begin{array}{c} \vec{w} \otimes \vec{x} \\ ??? \end{array}$$

$$A_{G\times H} = A_G \otimes I_{V_H} + I_{V_G} \otimes A_H$$

$$A_{G\times H}\left(\vec{\omega} \otimes \vec{x}\right)$$

$$= \left(A_G \otimes I_{V_H} + I_{V_G} \otimes A_H\right)\left(\vec{\omega} \otimes \vec{x}\right)$$

$$= \left(A_G \otimes I_{V_H}\right)\left(\vec{\omega} \otimes \vec{x}\right)$$

$$+ \left(I_{V_G} \otimes A_H\right)\left(\vec{\omega} \otimes \vec{x}\right)$$

$$\left(A_G \otimes I_{V_H}\right)\left(\vec{\omega} \otimes \vec{x}\right)$$

$$\overset{??}{=} \left(A_G \vec{\omega}\right) \otimes \left(I_{V_H} \vec{x}\right)$$

$$= (\lambda \vec{w}) \otimes (\vec{x}) = \underbrace{(2\vec{w}) \otimes \left(\tfrac{\lambda}{2}\vec{x}\right)}_{\text{has no direct relevance (side remark)}}$$

$$= \lambda (\vec{w} \otimes \vec{x})$$

Similarly

$$(I_{V_G} \otimes A_H)(\vec{w} \otimes \vec{x})$$

$$= \vec{w} \otimes (\mu \vec{x}) = \mu (\vec{w} \otimes \vec{x})$$

$$A_{G \times H}(\vec{w} \otimes \vec{x}) = (\lambda + \mu)(\vec{w} \otimes \vec{x})$$

Use this:

G, d-regular,

$$\lambda_n(A) \leq \dots \leq \lambda_2(A_G) \leq \lambda_1(A_G)$$

d regular

any vertex connected to d other vertices

$$A_G = \begin{bmatrix} & & \\ \dots 1 \dots 0 \dots 1 \; 0 & & \\ & & \\ \end{bmatrix}$$

any row

sum of entries

(row sum)

$= d$

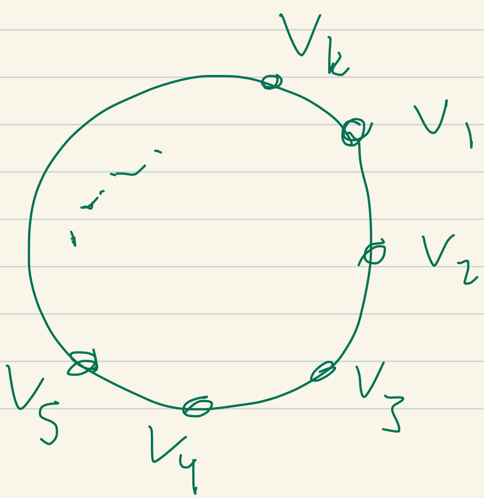
Claim: $\lambda_1(A_G) = d$

mult of d = # connected components

$\lambda_n(G) \geq -d$ and

mult of $-d$ = # of connected component that are bipartite

Convince you:



"cycle of length k"

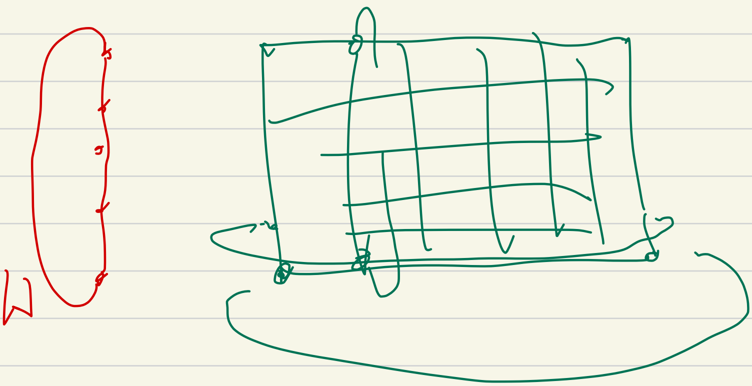
(a graph)

is not well interconnected graph

but for a 2-regular graph,

this is all you can do...

Grid graph, 4-regular

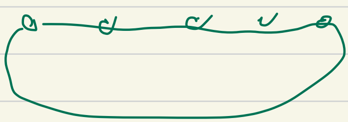


$G \times H$

is

4-regular

2-reg



G

2-reg