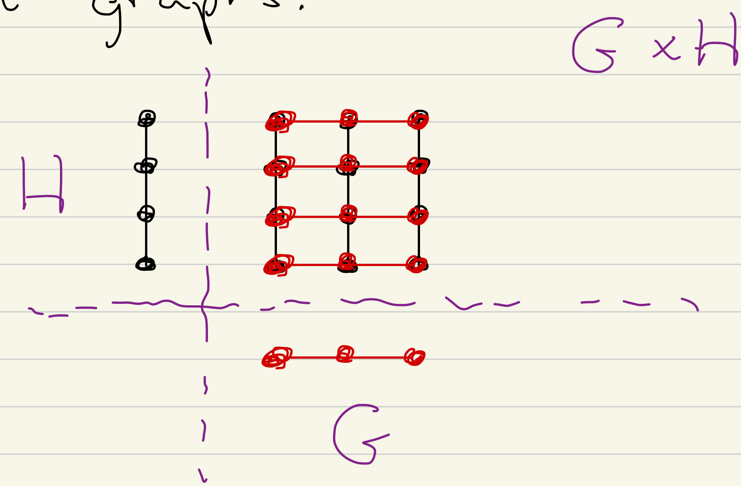
CPSC 536F Sept 24, 2025

- Definition: Cartesian product of graphs.



$$A_{G \times H} = ???$$

- Fibre product of graphs

$G \otimes H$, $A_{G \otimes H} = A_G \otimes A_H$

– "Good expansion" versus

"Bad expansion"

Back to kernels

Office hours this week:

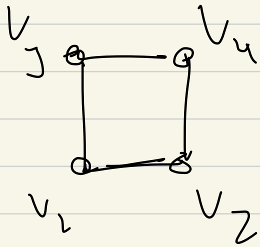
Wednesday 2–2:45 pm

Thursday 11am–12pm

Current Homework:

All in Appendices A & B

Example:



$$V = \{ v_1, v_2, v_3, v_4 \}$$

$$E = \{ \{v_1, v_2\}, \{v_3, v_4\}, \{v_1, v_3\}, \{v_2, v_4\} \}$$

Adjacency matrix of G

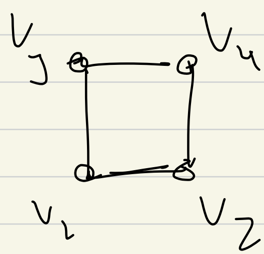
$$A_G = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

We say G graph $= (V_G, E_G)$

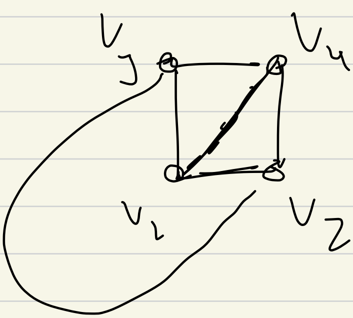
is d-regular if

$\deg_G(v) = d$ for all d

Example:

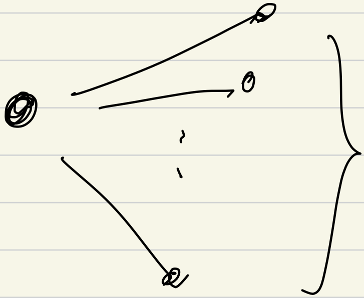


is 2-regular



is 3-regular

G is d-regular:

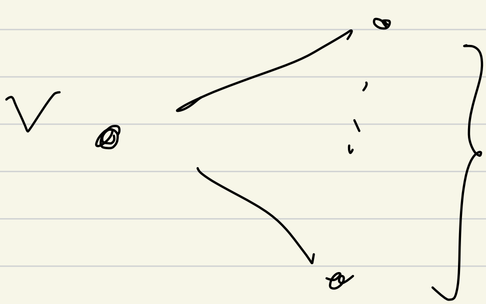


d incident edges

d adjacent vertices

A_G — row sum corresponding to vertex v

is $\deg_G(v)$



\# incident edges

$= \deg_G(v)$

What information does

A_G^k, $k = 1, 2, 3, \ldots$

give

$(A_G^k)(v, v') =$

of <u>walks</u> from v to v'

of length k,

walk! $v_0 - v_1 \longrightarrow v_2 \cdots - v_k$

<u>starts</u> at v <u>ends</u> at v'

Rem: If A_G is the adjacency matrix of a d-regular graph then we use

$$\lambda_n(G) \leq \dots \leq \lambda_2(G) \leq \lambda_1(G)$$

Claim:

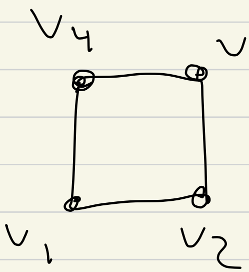
(1) $\lambda_1(G) = d$

$$A_G \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} d \\ \vdots \\ d \end{bmatrix} = d \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

(2) Multiplicity of d is # connected components

$\mathbb{B}^2$

Example:



$$A_G = \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

2-regular,

$$\lambda(A_G) =$$

$\mathbb{B}^1$

$$A_G = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \lambda\text{'s} = 1, -1$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

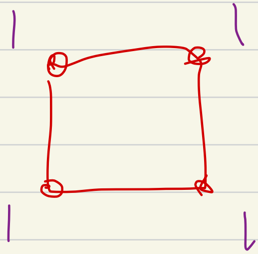
$$\lambda(\mathbb{B}^2) = \lambda(\mathbb{B}^1) + \lambda(\mathbb{B}^1)$$

$$= \{1, -1\} + \{1, -1\}$$

$$= \{2, 0, 0, -2\}$$

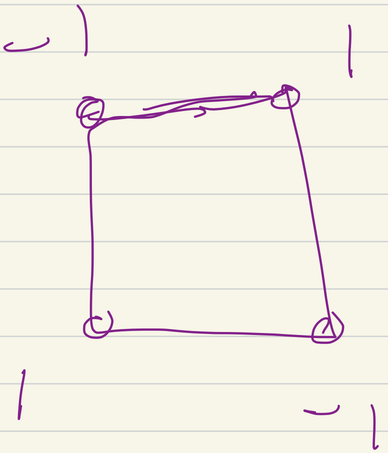
(add each pair of λ's of $(\mathbb{B}^1)$ multi-set)

$$\lambda = 2 \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$



$$\begin{bmatrix}1\\1\end{bmatrix} \otimes \begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{bmatrix}1\\-1\end{bmatrix} \otimes \begin{bmatrix}1\\-1\end{bmatrix}$$

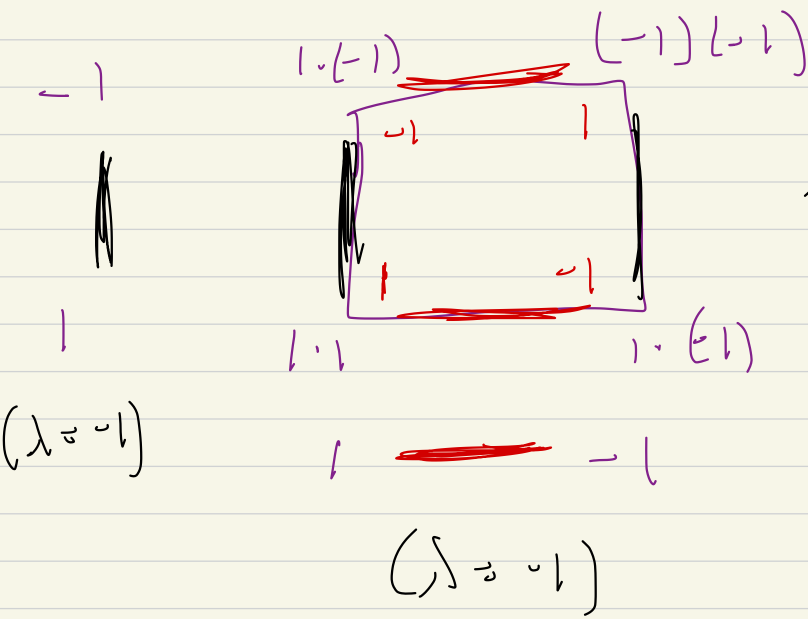


$$\begin{bmatrix}1\\ \leftrightarrows -1\\ 1\end{bmatrix}$$

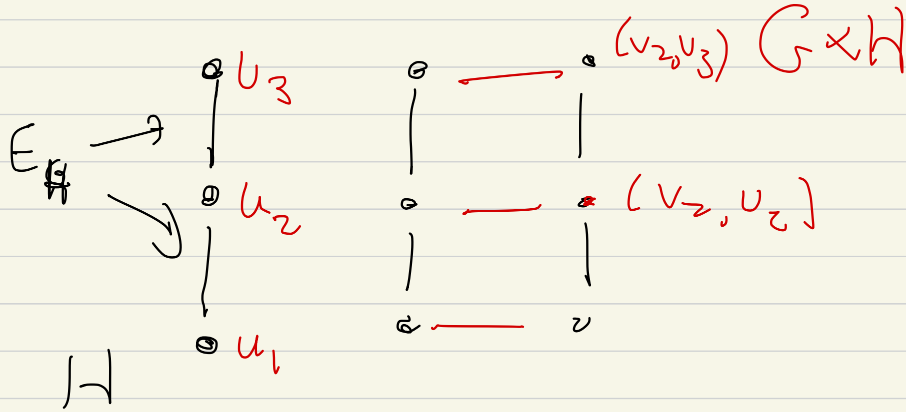
$\lambda = -2$

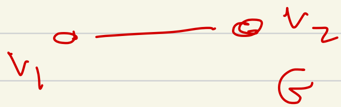
Claim: If A_G is d-regular, $\lambda_n \geq -d$, and $\lambda_n = -d$ iff G is

G is <u>bipartite</u>



$$\lambda = (-1) + (-1)$$
$$= -2$$





$A_H = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

A_G λ_i:

$1, -1$

<u>Def</u>: $G = (V_G, E_G)$, $H = (V_H, E_H)$

Then: $G \times H$ is the graph

$$V_{G \times H} = V_G \times V_H$$

$$E_{G \times H} = E_G \times V_H \cup V_G \times E_H$$

where

edge of G

vertex of H

$$e \in E_G \times V_H = \left(\{v, v'\}, u \right)$$

$$= \left\{ (v,u), (v',u) \right\}$$

and

$$e \in V_G \times E_H = \left(v, \{u, u'\} \right)$$

$$= \left\{ (v,u), (v,u') \right\}$$

Thm: $A_{G \times H} =$

$$\left(A_{G \otimes H} = A_G \otimes A_H \quad \cdots \right)$$

but

$$A_{G\times H} =$$

Example: $A_{G \times H}$:

$$
\begin{array}{c|cccccc}
 & v_1u_1 & v_1u_2 & v_1u_3 & v_2u_1 & v_2u_2 & v_2u_3 \\
\hline
(v_1,u_1) & 0 & 1 & 0 & 1 & 0 & 0 \\
(v_1,u_2) & 1 & 0 & 1 & 0 & 1 & 0 \\
(v_1,u_3) & 0 & 1 & 0 & 0 & 0 & 1 \\
(v_2,u_1) & 1 & & & & & \\
(v_2,u_2) & & 1 & & ? & ? & \\
(v_2,u_3) & & & 1 & & & \\
\hline
 & v_1u_1 & v_1u_2 & v_1u_3 & v_2u_1 & v_2u_2 & v_2u_3
\end{array}
$$