CPSC 536F, Sept 22, 2025

Last time:

- S, T sets: $\vec{x} \in \mathbb{R}^S$, $\vec{y} \in \mathbb{R}^T$:

$\mathbb{R}^S \otimes \mathbb{R}^T$ can be identified with

$\mathbb{R}^{S \times T}$, but this deceptive ...

$$\left[\begin{array}{l} \text{If } V, W \text{ are } \mathbb{R}\text{-vector spaces,} \\ \text{then } V \otimes W \text{ is an initial} \\ \text{element of the caterogy of} \\ \text{bilinear maps} \\ \qquad V \times W \longrightarrow Z \end{array}\right]$$

Only makes sense up to unique isom

$\bar{U}$ is an $\mathbb{R}$-vector space

(some abstract $\mathbb{R}$-vector spaces)

(concrete $\mathbb{R}^n$)

(1) A set, $\bar{U}$,

(2) Two rules:

$$(\alpha, u) \longmapsto \alpha u$$

$$\mathbb{R} \times \bar{U} \longrightarrow \bar{U}$$

"scalar multiplication"

$$(u_1, u_2) \longmapsto u_1 + u_2$$

$$\bar{U} \times \bar{U} \longrightarrow \bar{U}$$

"addition of two vectors"

s.t.

$$\gamma(\alpha u_1 + \beta u_2)$$
$$= (\gamma\alpha) u_1 + (\gamma\beta) u_2$$

$$(u_1 + u_2) + u_3 = u_1 + (u_2 + u_3)$$

$\vdots$

(really everything that $U = \mathbb{R}^n$ with familiar rules about scalar mult and addition)

Example:

$$\{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous} \}$$

e.g.,

$$3\sin(x) + 4e^x + (-12)x^2$$

is also continuous, since $\sin(x)$, e^x, x^2 are $\mathbb{R} \to \mathbb{R}$

Examples: $\mathbb{R}^n$, $\mathbb{R}^{m \times n} = m \times n$ matrices

$\mathbb{R}^m \otimes \mathbb{R}^n$

Point: $\mathbb{R}^m \otimes \mathbb{R}^n$ and $\mathbb{R}^{m \times n}$

"Usual" definition:

$$U \otimes W = \left\{ \sum \alpha_i \, u_i \otimes w_i \right\} \Big/ \sim$$

Where $\sim$ is (closure of)

$$\alpha \, u \otimes w = (\alpha u) \otimes w = u \otimes (\alpha w)$$

$$(u_1 + u_2) \otimes (w_1 + w_2) =$$

$$u_1 \otimes w_1 + u_1 \otimes w_2 + u_2 \otimes w_1 + u_2 \otimes w_2$$

$\mathbb{R}^{S \times T} \overset{\text{def}}{=}$ $S \times T$ matrices

$= \text{maps } S \times T \to \mathbb{R}$

So $\mathbb{R}^{S \times T}$ not canonically isom to $\mathbb{R}^S \otimes \mathbb{R}^T$

Say:

$\mathbb{R}^3$ has a basis

$\vec{e}_1 = (1,0,0)$, $\vec{e}_2 = (0,1,0)$ $\vec{e}_3 = (0,0,1)$

$\mathbb{R}^2$ has a basis

$\vec{e}_1 = (1,0)$, $\vec{e}_2 = (0,1)$

$\mathbb{R}^3$ contains $(1,2,3)$

$\mathbb{R}^2$ contains $(4,5)$

What is $\mathbb{R}^3 \otimes \mathbb{R}^2$?

$$\Big\{ \quad \vec{e}_1 \otimes \vec{e}_1 = (1,0,0) \otimes (1,0) ,$$

$$(1,2,3) \otimes (4,5) ,$$

$$(1,2,3) \otimes (4,4) ,$$

$$(21)\ (1,2,3) \otimes (4,4) ,$$

$$(21)\ (1,2,3) \otimes (4,4)$$
$$+ (37)\ (3,-1,0) \otimes (2,1)$$
$$+ (0)\ (1,0,0) \otimes (0,1) ,$$

$$\cdots \quad \Big\} \Big/ \sim$$

$$3 \cdot \left(7 \, (1,2,3) \otimes (4,4) \right),$$

$$21 \, (1,2,3) \otimes (4,4),$$

$$7 \cdot 3 \cdot 4 \, (1,2,3) \otimes (1,1),$$

generally finite sums

$$\boxed{\sum_{i=1}^{m} \alpha_i \, \vec{u_i} \otimes \vec{v_i}}$$

$\alpha_i \in \mathbb{R}$, $\vec{u_i} \in \mathbb{R}^3$, $\vec{v_i} \in \mathbb{R}^2$

$2 \cdot (1,2,3) \otimes (4,4) =$

$\underline{??}\ (1,0,0) \otimes (1,0)$ $+$

$\underline{??}\ (0,1,0) \otimes (1,0)$

$+$

$\underline{??}\ (0,0,1) \otimes (1,0)$

$+$

$\underline{??}\ (1,0,0) \otimes (0,1)$

$+$

$\underline{??}\ (0,1,0) \otimes (0,1)$

$+$

$\underline{??}\ (0,0,1) \otimes (0,1)$

Should have that $\mathbb{R}^3 \otimes \mathbb{R}^2$ has a basis $\vec{e}_1 \otimes \vec{e}_1, \vec{e}_2 \otimes \vec{e}_1, \ldots, \vec{e}_3 \otimes \vec{e}_2$

So in $\mathbb{R}^3 \otimes \mathbb{R}^2$

$(1,7,9) \otimes (1,2)$

"rank 1 tensor"

because ---

$$(1,7,9) \otimes (1,2)$$

$$= \left.\begin{array}{l} 1\ (1,0,0)\otimes(1,0)\\ 7\ (0,1,0)\otimes(1,0)\\ 9\\ 2\\ 14\\ 18 \end{array}\right\} \begin{bmatrix}1&7&9\\2&14&18\end{bmatrix}$$

rank 1

$$\begin{bmatrix}1&2\\7&14\\9&18\end{bmatrix}$$ rank 1

"Most elements of $\mathbb{R}^3 \otimes \mathbb{R}^2$

cannot be written as

rank 1 tensors
(matrices)

$\dim(U) = m$

$\dim(V) = n$

$\dim(U \otimes V) = mn$

Define: If $T \in U \otimes V$,

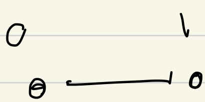
$\operatorname{rank}(T)$ is smallest r s.t.

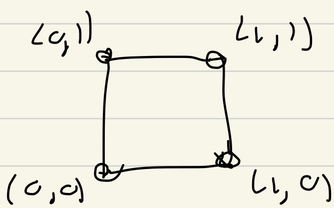
T can be written as

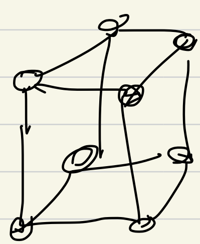
$$u_1 \otimes v_1 + u_2 \otimes v_2 + \cdots + u_r \otimes v_r$$

Thm: Rank of T is the same as rank of $\widetilde{T}$ where $\widetilde{T}$ is the matrix you get from T by ...

Say we look at :







$\mathbb{B}^1$ $\mathbb{B}^2$ $\mathbb{B}^3$

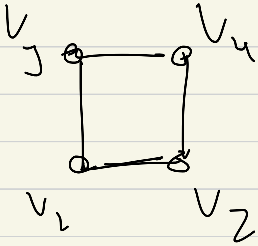
1-dim Boolean cube

What are the adjacency eigenvalues of $\mathbb{B}^3$?

A simple graph $G = (V_G, E_G)$ is

- a finite set, V_G
- $E_G \subset \binom{V_G}{2}$ = set of all unordered pairs of vertices.

Example:



$$V = \{ v_1, v_2, v_3, v_4 \}$$

$$E = \{ \{v_1, v_2\}, \{v_3, v_4\}, \{v_1, v_3\}, \{v_2, v_4\} \}$$

Adjacency matrix of G

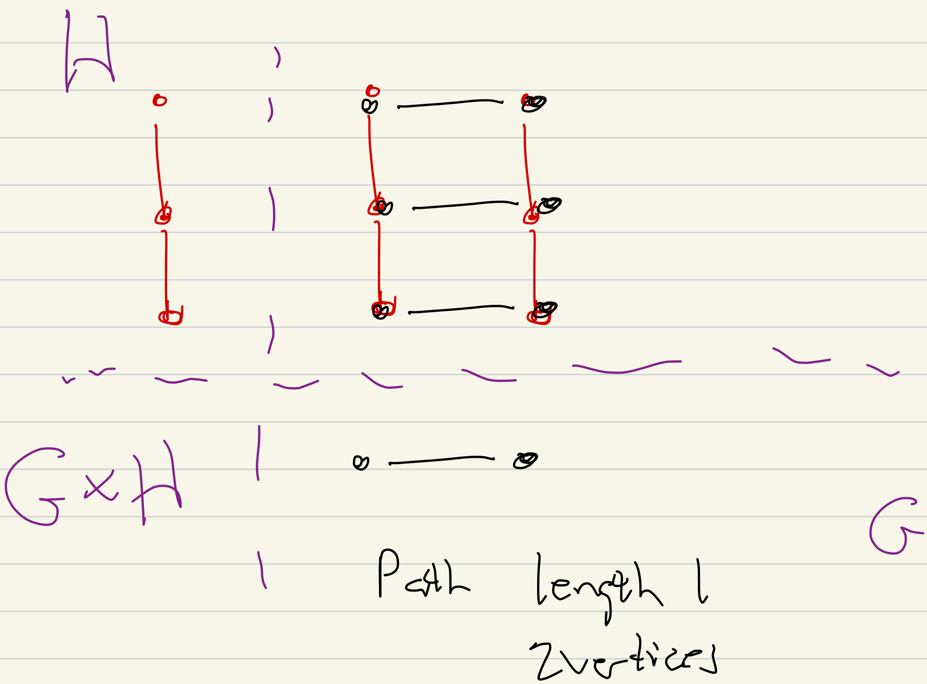
$$A_G = \begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$$A_G(v, v') = \begin{cases} 1 & \text{if } \{v, v'\} \in E_G \\ 0 & \text{if not} \end{cases}$$

A_G is symmetric.

If G, H graphs then

$G \times H$ Cartesian product



$V(G \times H) = V(G) \times V(H)$

$E(G \times H) = \ldots$

$$A_{G \times H}$$

$$= \quad ???$$

But given the standard basis

$\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ of $\mathbb{R}^n$

Rem: $\vec{e}_1$ in $\mathbb{R}^2$ is $(1,0)$

$\vec{e}_1$ in $\mathbb{R}^3$ is $(1,0,0)$

so "standard basis" assumes you know what is the ambient space...

$$A \in \mathbb{R}^{m \times n}, \quad B \in \mathbb{R}^{p \times q}$$

$$(A \otimes B)(\vec{u} \otimes \vec{v})$$

$$\overset{\text{def}}{=\!=} (A\vec{u}) \otimes (B\vec{v})$$

extend linearly

$$(A \otimes B)(\vec{u}_1 \otimes \vec{v}_1 + \vec{u}_2 \otimes \vec{v}_2)$$

$$A\vec{u}_1 \otimes B\vec{v}_1 + A\vec{u}_2 \otimes B\vec{v}_2$$

What makes things "work well"?

Graph theory: