CPSC 536F Sept. 19, 2025

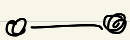
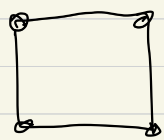
H_2 = Standard 2×2 Hadamard matrix

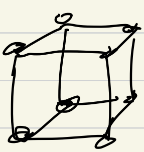
$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Kronecker Product, Tensor Product,

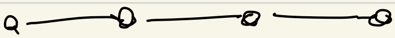
$$H_4 = \left[\begin{array}{c|c} H_2 & H_2 \\ \hline H_2 & -H_2 \end{array}\right] = H_2 \otimes H_2$$

$=$

$\mathbb{B}^1 =$  , $\mathbb{B}^2 =$ 

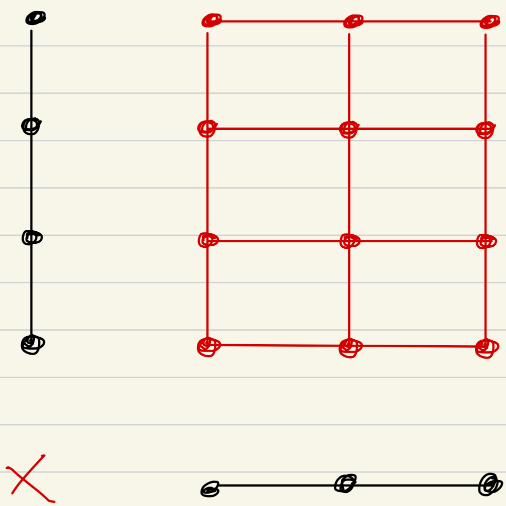
$\mathbb{B}^3 =$  $= \mathbb{B}^1 \times \mathbb{B}^2$

Want to talk about graphs:

Path 

3 edges, 4 vertices

3x4 grid



Cartesian Product

Examples of kernels

$$\mathcal{X} \times \mathcal{X} \to \mathbb{R}$$

① $\Phi : \mathcal{X} \to \mathbb{R}^n$ (any map of sets)

then $k(x,x') \stackrel{\text{def}}{=} \Phi(x) \cdot \Phi(x')$

is positive semi-definite

$\left[\text{e.g. } \mathcal{X} = \mathbb{R}^2,\ x, x' \mapsto (1 + x \cdot x')^2\right]$

①' Not definite if $|\mathcal{X}| > n$

② Sum of kernels, via ⊕

③ Product of kernels, via ⊗

(4) "Kernels representable in $\mathbb{R}^n$"

Admin:

- Hand in some homework around Sept 30, 2025
- I'll give you feedback, you can resubmit them at the end of the course
- Some problems may be optional
- See me if you'd like to do a project/presentation

Current document:

App A: Exercises on kernel trick

App B: " in linear alg

App C:

⋮

Glossary

Office hours: start next week

$$H_2 \cdot H_2$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= 2I$$

$\Rightarrow$ (H_2 is symm, hence diagonalizable, real eigenvalues)

only possible: $H_2 \vec{v} = \lambda \vec{v}$

$$H_2 H_2 \vec{v} = \lambda^2 \vec{v}$$

$$2I \vec{v} = \lambda^2 \vec{v}$$

$\Rightarrow \lambda = \pm\sqrt{2}$.

Rem: Here is how we know that H has one eigenvalue $\sqrt{2}$ and one $-\sqrt{2}$:
if not, then

$$H_2 = U \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} U^{-1}$$

and we must have $\lambda_1 = \lambda_2 = \lambda$.

But then

$$H_2 = U (\lambda I) U^{-1} = \lambda I$$

which is impossible. Hence

So λ's H_2 $\begin{cases} \sqrt{2} \\ -\sqrt{2} \end{cases}$ $\quad H_2^2 = 2I$

$$H_4 \overset{def}{=} H_2 * H_2$$

$$H_2 \quad \begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix}$$

$$= \left[\begin{array}{c|c} H_2 & H_2 \\ \hline H_2 & -H_2 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ \hline 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{array}\right] \in \mathbb{R}^{4\times 4}$$

$$H_4^2 = (H_2 * H_2)(H_2 * H_2)$$

$$(H_2 \cdot H_2) * (H_2 \cdot H_2) = (2I) * (2I)$$

If A is $m \times n$

B is $p \times q$

$A \otimes B$ is $(mp) \times (nq)$

def (?) $\begin{bmatrix} Ba_{11} & Ba_{12} & \cdots & Ba_{1n} \\ \vdots & & & \vdots \\ Ba_{m1} & \cdots & & Ba_{mn} \end{bmatrix}$

as block matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

A $m \times n$ matrix

$\vec{x} \in \mathbb{R}^n$ $\quad A\vec{x} \in \mathbb{R}^m$

B $p \times q$ matrix

$\vec{y} \in \mathbb{R}^q$, $\quad B\vec{y} \in \mathbb{R}^p$

$$(A * B)\left(\overset{\mathbb{R}^n}{\vec{x}} * \overset{\mathbb{R}^q}{\vec{y}}\right)$$

$$= \underset{\mathbb{R}^m}{(A\vec{x})} * \underset{\mathbb{R}^p}{(B\vec{y})}$$

??

$\vec{x} * \vec{y} =$

$(x_1, \dots, x_n) * (y_1, \dots, y_q)$

$= (x_1 y_1, \quad \dots \quad, x_n y_q)$

all nq possible combos

2 orders make the most sense

$(x_1 y_1, x_1 y_2, \dots x_1 y_q, x_2 y_1, \dots)$

$(x_1 y_1, x_2 y_1, x_3 y_1, \dots$

$(x_1 y_1, x_1 y_3, x_2 y_4, \dots)$

$\mathbb{R}^n = ?$

$\mathbb{R}^S$, S is a set,

$\mathbb{R}^S$: maps $S \to \mathbb{R}$

$\mathbb{R}^S * \mathbb{R}^T$ maps $S \times T \to \mathbb{R}$

$\mathbb{R}^n = \mathbb{R}^{[n]} = \mathbb{R}^{\{1,2,\dots,n\}}$

$\mathbb{R}^n * \mathbb{R}^q \longrightarrow \mathbb{R}^{\{1,\dots,n\} \times \{1,\dots,q\}}$

$\cong \mathbb{R}^{nq}$

$H_2 \otimes H_2$ eigenvalues

$$\begin{Bmatrix} +\sqrt{2}, \\ -\sqrt{2} \end{Bmatrix} \quad \begin{Bmatrix} +\sqrt{2}, \\ -\sqrt{2} \end{Bmatrix}$$

A, B square matrices, diff dims

$$A\vec{v} = \lambda\vec{v}, \quad B\vec{w} = \mu\vec{w}$$

$$(A \otimes B)(\vec{v} \otimes \vec{w}) = (A\vec{v}) \otimes (B\vec{w})$$

good news

$$= (\lambda\vec{v}) \otimes (\mu\vec{w})$$

$$= \lambda\mu(\vec{v} * \vec{w})$$

Also: if $\vec{v}_1 \perp \vec{v}_2$, ω_1, ω_2 anything

$$(\vec{v}_1 * \vec{w}_1) \cdot (\vec{v}_2 * \vec{w}_2)$$

$$= (\vec{v}_1 * \vec{w}_1)^T \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \vec{v}_2 * \vec{w}_2$$

$$=$$

$$(x_1, \dots, x_n) \cdot (y_1, \dots, y_n)$$

$$[x_1 \cdots x_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = [x_1 \cdots x_n]\, I \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

Application:

$$\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^6$$

$$(x,y) \longmapsto (1, \sqrt{2}x, \sqrt{2}y, x^2, \sqrt{2}xy, y^2)$$

Define: $k : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

$$k((x_1,y_1),(x_2,y_2))$$

$$= \Phi(x_1,y_1) \cdot \Phi(x_2,y_2)$$

Claim: This is positive semidefinite

If $\mathcal{X}$ is any set,

$$\Phi : \mathcal{X} \to \mathbb{R}^m$$

then

$$k(x, x') \overset{\text{def}}{=}$$

$$\Phi(x) \cdot \Phi(x'),$$

then k is a positive semidefinite kernel function.

We know $\mathcal{L} : \mathbb{R}^n \to \mathbb{R}^m$ s.t.

$$\mathcal{L} = M^T M, \text{ then } \mathcal{L} \text{ pos def}$$

~~Fact~~: If

k, k' are pos semi ~~def~~

on $\mathcal{X} \times \mathcal{X} \to \mathbb{R}$,

then

$k \cdot k'$ is also

positive semi definite

(using $\ast$)