CPSC 536F Sept. 17, 2025

More examples:

Circulant matrices

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ & & \ddots & \\ a_2 & a_3 & & a_1 \end{bmatrix}$$

$$K = \frac{K + K^T}{2} + \frac{K - K^T}{2}$$

$$H_2 \quad \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad H_4 = H_2 * H_2$$

kernels $\mathcal{X} \times \mathcal{X} \to \mathbb{R}$

If $\Phi : \mathcal{X} \underbrace{\longrightarrow}_{\text{set theoretic}} \mathbb{R}^n$

then

$$k(x, x') = \Phi(x) \cdot \Phi(x')$$

is positive semidefinite, not definite when $|\mathcal{X}| > n$.

$$\mathcal{X} \longrightarrow \mathbb{R}^{\mathcal{X}} \quad \text{so}$$

$\Phi : \mathbb{R}^{\mathcal{X}} \to \mathbb{R}^n$ is a linear map.

Today : I'll send out a when2meet for regular office hours.

If you can't make them:

To: jf@cs.ubc.ca

Subject: ... 536 ...

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Homework problems starting to appear: ellipses, shell, level-set averages, ... .

- Also some on matrices

# Circulant matrices

$$K = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ & & \ddots & \\ a_2 & a_3 & & a_1 \end{bmatrix}$$

$a_1, \ldots, a_n \in \mathbb{R}, \mathbb{C}$

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General Remark:

If $K$ is any $\mathbb{R}^{n \times n}$

$$K = \underbrace{\frac{K + K^T}{2}}_{K_{symm}} + \underbrace{\frac{K - K^T}{2}}_{K_{antisym}}$$

$$\left(\frac{K+K^T}{2}\right)^T = \frac{K^T+K^{TT}}{2} = \frac{K^T+K}{2}$$

$$= \frac{K+K^T}{2}$$

$K_{symm}$ : $K_{symm}^T = K_{symm}$

so $K_{symm}$ is symmetric.

Similarly

$$K_{antisymm}^T = -K_{antisymm}.$$

We say $A \in \mathbb{R}^{n\times n}$ is <u>antisymmetric</u>

if $A^T = -A$.

Say $K = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$

Then if $\omega^3 = 1$

$$\underbrace{\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix} \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix}}_{\begin{bmatrix} a+b\omega+c\omega^2 \\ c+a\omega+b\omega^2 \\ \\ \end{bmatrix}} = \underbrace{\left(a+b\omega+c\omega^2\right) \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix}}_{\begin{bmatrix} a+b\omega+c\omega^2 \\ a\omega+b\omega^2+c\omega^3 \\ \\ \end{bmatrix}}$$

$\omega^3 = 1 \iff$ Cyclic group of order 3 $\iff \mathbb{Z}/3\mathbb{Z}$

e.g. $K = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ has

eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \perp$ to $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 - 1 = 0$$

Claim: If $K^T = K$, $K \in \mathbb{R}^{n \times n}$

and

$$K\vec{u} = \lambda \vec{u}, \quad K\vec{v} = \mu \vec{v}$$

and $\lambda \neq \mu$, then $\vec{u} \cdot \vec{v} = 0$

$$\vec{u}^{T} K \vec{v} = \vec{u}^{T} \mu \vec{v} = \mu\left(\vec{u}^{T} \vec{v}\right)$$

$$\left(K^{T} \vec{u}\right)^{T} \vec{v} \qquad = \mu\left(\vec{u} \cdot \vec{v}\right)$$

$$= \left(K \vec{u}\right)^{T} \vec{v} \qquad \longleftarrow \quad K^{T} = K$$

$$= \left(\lambda \vec{u}\right)^{T} \vec{v}$$

$$= \lambda \vec{u}^{T} \vec{v}$$

$$= \lambda\left(\vec{u} \cdot \vec{v}\right)$$

So

$$(\lambda - \mu)\left(\vec{u} \cdot \vec{v}\right) = 0 \qquad (!!)$$

$$K = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$$

$$K = K^T \iff b = c$$

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$M \in \mathbb{R}^{2\times2}$ s.t. the eigenvalues

are $7, \; 7 + 0.001$

want

$$MU = U \begin{bmatrix} 7 & 0 \\ 0 & 7.001 \end{bmatrix}$$

any invertible $U$ will work

$$U = \begin{bmatrix} 2025 & 2025 \\ 536 & 536.0001 \end{bmatrix}$$

$U$ is invertible

$$M = U \begin{bmatrix} 7 & 0 \\ 0 & 7.0001 \end{bmatrix} U^{-1}$$

really bad...

$$= U \left( 7I + \begin{bmatrix} 0 & 0 \\ 0 & 0.0001 \end{bmatrix} \right) U^{-1}$$

$$= 7I + U \begin{bmatrix} 0 & 0 \\ 0 & 0.0001 \end{bmatrix} U^{-1}$$

$$M \begin{bmatrix} 2025 \\ 536 \end{bmatrix} = 7 \begin{bmatrix} 2025 \\ 538 \end{bmatrix}$$

$$M \begin{bmatrix} 2025 \\ 536.00001 \end{bmatrix} = 7.0001 \begin{bmatrix} 2025 \\ 536.0001 \end{bmatrix}$$

$$\cos\left( \begin{bmatrix} 2025 \\ 536 \end{bmatrix}, \begin{bmatrix} 2025 \\ 536.0001 \end{bmatrix} \right)$$

almost $= 1$

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Eigenvalues of $7I$ : $7, 7$

Eigenvectors of $\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$ eig $7$ : $\mathbb{R}^2$

Question: Find an orthonormal eigenbasis for

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Ans: $\lambda = 3$, $\lambda = 0, 0$

Eigenvectors with eigenvalue 0:

$$\ker\left( \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} - \lambda I \right)$$

$\lambda = 0$

$$= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; x + y + z = 0 \right\}$$

$$x = -y - z$$

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$y = 1$

$z = 0$

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

$z = 1$

$y = 0$

$$\operatorname{proj} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ onto } \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}^{\perp}$$

$$\longrightarrow \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{pmatrix}$$

-- or --

$$\begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix}, \begin{bmatrix} 1 \\ \omega^2 \\ \omega \end{bmatrix}$$

$\omega^3 = 1$

$\omega \neq 1$

Rem: If $\vec{u}, \vec{v} \in \mathbb{C}^n$

$$\vec{u} \cdot \vec{v} = \begin{cases} u_1 \bar{v}_1 + u_2 \bar{v}_2 + \dots + u_n \bar{v}_n \\ \text{OR} \\ \cancel{\bar{u}_1 v_1 + \bar{u}_2 v_2 + \dots + \bar{u}_n v_n} \end{cases}$$

where $\overline{\alpha + i\beta} = \alpha - i\beta$

$\alpha, \beta \in \mathbb{R}$

$$(i\vec{u}) \cdot \vec{v} = i(\vec{u} \cdot \vec{v})$$

$$\vec{u} \cdot (i\vec{v}) = \bar{i}(\vec{u} \cdot \vec{v}) = -i(\vec{u} \cdot \vec{v})$$

We define

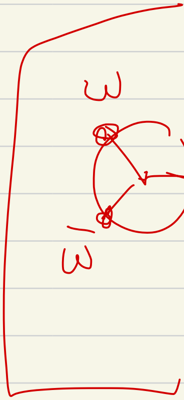
$$A^H \overset{def}{=} \overline{A}^T$$

$\overline{A}$ we conjugate all entries

$$\vec{u} \cdot \vec{v} = \vec{u}\left(\vec{v}^H\right)$$

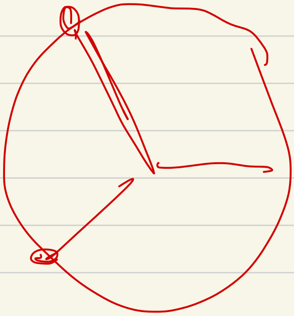
$$(\alpha \vec{u}) \cdot \vec{v} = \alpha\, \vec{u} \cdot \vec{v}$$

$$\vec{u} \cdot (\alpha \vec{v}) = \overline{\alpha}\, \vec{u} \cdot \vec{v}$$




$$\begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \omega^2 \\ \omega \end{bmatrix} = \begin{bmatrix} 1 \\ \omega \\ \overline{\omega} \end{bmatrix} \begin{bmatrix} 1 \\ \overline{\omega} \\ \omega \end{bmatrix} = 1 + \cancel{\omega\overline{\omega}} + \cancel{\overline{\omega}\omega}$$

$$\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$



$$\omega^2 = \overline{\omega} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

$$\omega^3 = 1$$

$$\omega \neq 1$$

$$\begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \omega^2 \\ \omega \end{bmatrix}$$

$$= 1 + \omega\,\overline{\omega^2} + \omega^2\,\overline{\omega}$$

$$= 1 + \omega\,\omega^4 + \omega^2\,\omega^2$$

$$= 1 + \omega^5 + \omega^4 = 1 + \omega^2 + \omega = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\boxed{\begin{aligned} &\omega^3 = 1 \\ &\omega \neq 1 \\ &\omega = e^{2\pi i/3} \end{aligned}}$$

$\lambda = 3$ $\lambda = 0$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix}, \begin{bmatrix} 1 \\ \omega^2 \\ \omega \end{bmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix} = \begin{bmatrix} 1+\omega+\omega^2 \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

CLASS ENDS

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$ any 4x4 circulant

eigenvectors:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix}$$

$$\left( K = \sum \vec{v}_j \vec{v}_j^{\,T} \lambda_j \right)$$

$$\left( \cdot \; - \; \vec{v}_3 \, \vec{v}_2^{\,T} \; - \; \cdot \right) \cdot \vec{v}_2^{\,T}$$

$$\vec{v}_3$$

HW: Prove iff circulant matrices (anonymous)

$$\vec{y}' = A\vec{y} \quad , \quad \vec{y}(0) = \vec{y}_0 .$$

$$\vec{y}(t) = \underbrace{e^{At}}_{\left(I + At + A^2 t^2/2 + \cdots\right)} \vec{y}_0$$