CPSC 536F Sept 15, 2025

For $K \in \mathbb{R}^{n \times n}$, the following are equivalent

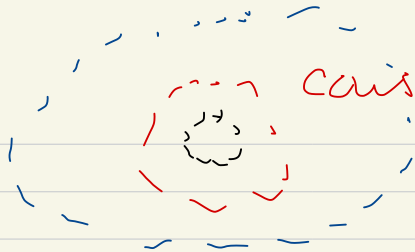
① $K^T = K$ (K is symmetric)

② $$KU = U \begin{bmatrix} \lambda_1 & & 0\text{'s} \\ & \lambda_2 & \\ 0\text{'s} & & \ddots \\ & & & \lambda_n \end{bmatrix}$$

$$= U \Lambda$$

($U^T = U^{-1}$,
U orthonormal columns)

where $\lambda_i \in \mathbb{R}$, U is an orthogonal matrix. We haven't yet proved this.



Today:

- Homework: the circle phenomenon also works for ellipses! (In a sense)

- Examples of positive (semi) definite matrices

- Thm: $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$

given by

$k(x, x') = \Phi(x) \cdot \Phi(x')$,

$\Phi : \mathcal{X} \to \mathbb{R}^n$ is not pos. def.

if $|\mathcal{X}| > n$.

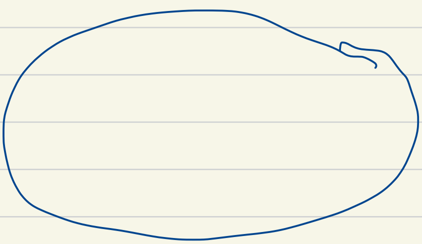
$(x,y) \in \mathbb{R}^2$

What $\mathbb{E}[x^2]$ on the ellipse

$$\left\{\begin{array}{l} x = a\cos\theta \\ y = b\sin\theta \end{array}\right\} \left\{ \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \right.$$

ChatGPT:

$\mathbb{E}$ according to arc length



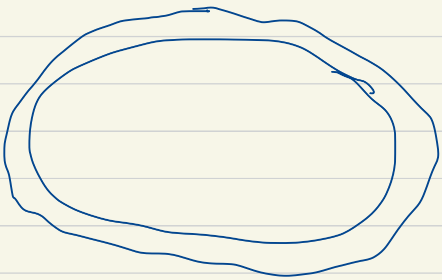
$$(ds)^2 = (dx)^2 + (dy)^2$$

$$\mathbb{E}[x^2] = \text{---} \quad \underbrace{K(e),\ E(e)}_{\text{elliptic functions}}$$

Now I want

$$\left(x^2/a^2\right) + \left(y^2/b^2\right)$$ between

1 and $1+\varepsilon$

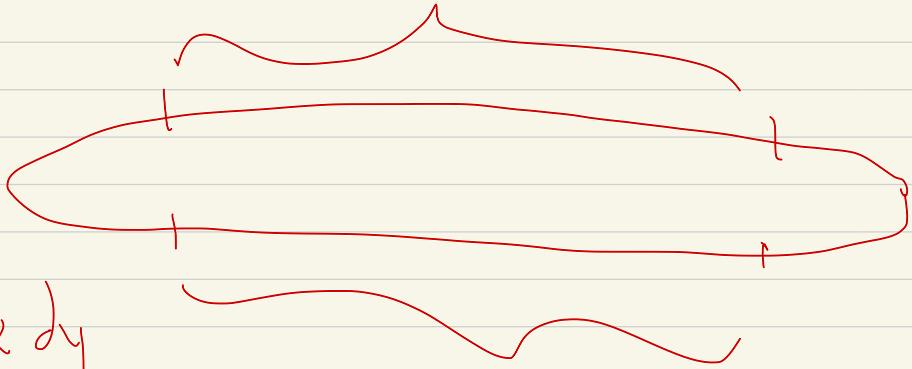


"shell"

then

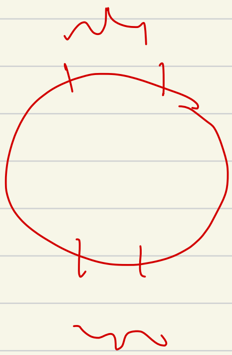
$$E[x^2] = \frac{a^2}{2}$$

Isn't $d\left(\frac{x}{a}\right) d\left(\frac{y}{b}\right) = \frac{1}{ab}\, dx\, dy$?



$dx\,dy$

$d\left(\frac{x}{a}\right) d\left(\frac{y}{b}\right)$



Examples of symmetric matrices:

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}, \quad [a]$$

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & & & \vdots \\ 1 & \cdots & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \qquad \mathbb{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$= \left(\mathbb{1}/\sqrt{n} \right) \left(\mathbb{1}/\sqrt{n} \right)^T n$$

$$\Rightarrow \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$$

has eigenvalues

$$n = \lambda_1 \geq \lambda_2 = \cdots = \lambda_n = 0$$

indeed

$$\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \\ 1 & \cdots & 1 \end{bmatrix} = \sum_{j=1}^{n} (\vec{v}_j)(\vec{v}_j)^T \cdot \lambda_j$$

where

$$\vec{v}_1 = 1/\sqrt{n}, \quad \lambda_1 = n, \quad \cdots$$

Question: What if $K = K^T$, $K \in \mathbb{R}^{n \times n}$

$$\vec{v}_1 = \mathbb{1}/\sqrt{n}, \quad \lambda_1 = n$$

$\lambda_2, \ldots, \lambda_n$ very small

$$|\lambda_2|, \ldots, |\lambda_n| \underbrace{\ll}_{\text{much smaller}} n$$

then

$$M = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix} + \mathcal{E}$$

$$\mathcal{E} = \sum_{j=2}^{n} (\vec{v}_j)(\vec{v}_j)^T \lambda_j \quad \text{"small"}$$

So K has a very good rank 1 approximation

(If $K = K^T$, then

$$\operatorname{rank}(K) = \dim(\operatorname{Image}(K))$$

$$= \_\_\_\_$$

$$= \#\{\text{non-zero eigenvalues}\}$$

In picture, low rank approx

K symm $\Longleftrightarrow$

$$KU = U\Lambda, \quad \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$U = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$$

$\vec{v}_1, \ldots, \vec{v}_n$ orthonormal eigenbasis

$$\Longleftrightarrow \quad K = \sum_{j=1}^{n} \vec{v}_j (\vec{v}_j^{\,T}) \lambda_j$$

K is positive semidefinite

iff $\lambda_j \geq 0$ for all j

K is positive definite iff $\lambda_j > 0$ for all j.

Example:

$K \in \mathbb{R}^{n\times n}$, $\quad M = \mathbb{R}^{m\times n}$,

$$K = M^T M,$$

then K is positive semidefinite.

Proof: $\underbrace{\vec{u} \cdot (K\vec{u})}_{\in \mathbb{R}} = \underbrace{\vec{u}^T (K\vec{u})}_{\mathbb{R}^{1\times 1}}$

$$= \vec{u}^T K \vec{u}$$

$$= \vec{u}^T M^T M \vec{u}$$

$$= (M\vec{u})^T (M\vec{u})$$

$$= (M\vec{u}) \cdot (M\vec{u}) \geq 0$$

for all $\vec{u} \in \mathbb{R}^n$.

Rem: To get the SVD (singular-value decomposition) of $M \in \mathbb{R}^{m \times n}$, we let $K = M^T M$ and find eigenpairs (eigenvalues/vectors) of K.

<u>Rem:</u> K is positive semidefinite iff $K = M^T M$ for some

$M \in \mathbb{R}^{m \times n}$, for some m.

e.g.,

$$K = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \\ 1 & & 1 \end{bmatrix},$$

$$K = M^T M \text{ where } \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}$$

$$= \left(\mathbb{1}/\sqrt{n}\right)\left(\mathbb{1}/\sqrt{n}\right)^T n$$

Similarly for any symmetric K,

using $\vec{v}_1, \dots, \vec{v}_n$

$\lambda_1, \dots, \lambda_n$

$$= \left(\vec{v}_1 \sqrt{\lambda_1}\right)^T \left(\vec{v}_1 \sqrt{\lambda_1}\right)$$

$$+ \left(\vec{v}_2 \sqrt{\lambda_2}\right)^T \left(\vec{v}_2 \sqrt{\lambda_2}\right)$$

$$+ \dots$$

Observation: Say that

$K = M^T M$, and $M \in \mathbb{R}^{m \times n}$

$m < n$

e.g.,

$$K = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}, \quad M^T = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \quad M = [1 \cdots 1]$$

Can K be pos. def., i.e.

$$\vec{u} \cdot (K\vec{u}) = \vec{u}^T K \vec{u} > 0$$

whenever $\vec{u} \neq \vec{0}$?

$$\ker([1 \cdots 1])$$

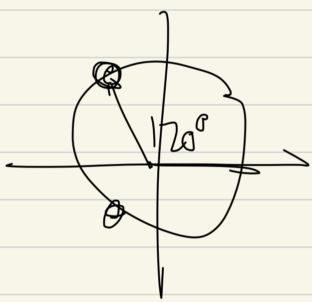
$$= \left\{ \vec{u} \;\middle|\; [1 \cdots 1] \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = 0 \right\}$$

$$= \{ \vec{u} \mid u_1 + \cdots + u_n = 0 \}$$

So ... similar...

$$\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = (a+b+c) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\left.\begin{array}{l} \omega^3 = 1, \quad \omega \neq 1, \\ \text{so } (\omega^3 - 1) = (\omega - 1)(\omega^2 + \omega + 1) \end{array}\right\} \text{omega}$$



$$\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix} \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix}$$

$$= \begin{pmatrix} a + b\omega + \\ c\omega^2 \end{pmatrix} \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix}$$

"circulant"

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & a_2 \cdots & a_{n-1} \\ & & a_1 & \\ & & & \ddots \end{pmatrix}$$

"circulant"

Homework on circulant.